ON DIRICHLET, PONCELET AND ABEL PROBLEMS

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Abstract. We offer connections between some problems of PDE, geometry, algebra, analysis and physics. The uniqueness of the solution of the Dirichlet problem and some another boundary value problems for the string equation inside of an arbitrary biquadratic algebraic curve is considered. It is shown that the solution is non-unique if and only if corresponding the Poncelet problem for two conics has a periodic trajectory. Similarly some other problems are proved to be equivalent to it. Among them there are the solvability problem of the algebraic Pell-Abel equation and an indeterminacy problem of a moment problem that generalizes well-known trigonometrical moment problem. Solvability criterions of above problems can be presented in form $\theta \in \mathbb{Q}$ where the number $\theta = m/n$ is connected with the concrete problem data. We demonstrate also an intimate relation of the above mentioned problems with such problems of the modern mathematical physics as elliptic solutions of the Toda chain, static solutions of the classical Heisenberg XY-chain and biorthogonal rational functions on elliptic grids in the theory of the Padé interpolation.

Keywords: biquadratic curve; Dirichlet problem; Neumann problem; string equation; moment problem; Poncelet problem; Pell-Abel equation; Toda chain; Heisenberg chain; biorthogonal rational functions
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Contents

1. Introduction

This work is devoted to new connections between some problems of mathematics such as some ill-posed boundary value problems in a bounded semialgebraic domain for partial differential equations, a moment problem, the Poncelet problem of the projective geometry, the algebraic Pell-Abel equation and some other problems, recently revealed by authors.

Study of ill-posed boundary value problems in bounded domains for partial differential equations go back to J.Hadamard and they are a popular object of modern infestigations (see s. 2.1). In this paper we will examine generally the Dirichlet problem for the string equation

$$u_{xy} = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u|_C = \phi \quad \text{on} \quad C = \partial\Omega, \tag{1.2}$$

the solution uniqueness for which is the problem of existence of a nontrivial solution of the homogeneous Dirichlet problem

$$u|_C = 0 \quad \text{on} \quad C. \tag{1.3}$$

The functions u, ϕ are assumed to be complex-valued functions of real variables. We will consider this and other boundary value problems in semialgebraic domains, the boundary of which is given by so-called biquadratic algebraic curve

$$F(x, y) := \sum_{i,k=0}^{2} a_{ik} x^i y^k = 0 \tag{1.4}$$

where x^i, y^k are powers. We will consider canonical forms of the curve (1.4) to which the generic curve can be transformed by linear-fractional replacements and we will give criterions of uniqueness breakdown in the form

$$\tau \in \mathbb{Q} \tag{1.5}$$

where the number τ is determined by the curve C. Our investigations are based on a study of the John mapping generated on C by characteristics of the equation (1.1), see propositions 29 (s. 6) and 19 (s. 3.5) where we use that this John mapping becomes an usual shift after a transform on universal covering group of the variety (1.4).

We will have observed a remarkable connection of this problem with the Poncelet problem. The Poncelet problem is one of famous problems of projective geometry and it by itself has numerous links with a set of different problems of analysis and physics (see [13],[10] [26] and below s. 4.1). We will prove that for generic biquadratic curve C the Dirichlet problem has a non-unique solution if and only if corresponding the Poncelet problem has a periodic trajectory (proposition 25, s. 4.5). And therefore we will give a new criterion of periodicity in the form (1.5) that differs from the well-known Cayley criterion (see s. 4.6).

Remind that the existence of a periodic trajectory in the Poncelet problem means that each trajectory is periodic by the big Poncelet theorem [13]. Note that different cases of disposition of conics give different cases of curves C and a new setting of the Dirichlet problem (1.1),(1.3) for unbounded domains is more suitable than the classical setting (see s. 2.2).

We will observe one more remarkable connection of the Poncelet problem with the solvability problem of algerbraic the Pell-Abel equation

$$P(t)^2 - R(t)Q(t)^2 = L \tag{1.6}$$

where for given polynomial $R(t)$ of one variable t one should find polynomials P, Q and a constant L satisfying the equation. The Pell-Abel equation solvability is one of famous problems also and this algebraic problem by itself has numerous connections with many problems of analysis (see below s. 5).

We will prove (see proposition 27 of s. 5.2) that the Pell-Abel equation (5.1) with a polynomial R of the third or fourth order has a solution if and only if corresponding the Poncelet problem has a periodic trajectory with an even period. On this way we obtain a new criterion in the form (1.5) which differs from well-known others, for example, from well-known Zolotarev's porcupine (see [43]).

We will show that the same condition (1.5) is a criterion of solution uniqueness to within an additive constant of the Neumann problem

$$u_{\nu_*}|_C = \psi \tag{1.7}$$

for the same equation (1.1) in the same domain where u_{ν_*} is a derivative with respect to the conormal ν_* (statement 3 of s. 2.3). We will show also that the same condition is a criterion of indeterminacy of a moment problem on the curve C (statement 3 of s. 2.3). By means of the duality equation-domain we will obtain an equivalent problem (2.22) in the form of a hyperbolic (in some cases of corresponding Poncelet problem) equation of the fourth order with only two boundary data on characteristics instead four as in the boundary value problem of the Goursat type, where we will have solution uniqueness almost always.

We will have observed some connections with a problem on classical XY-spin chains and a problem from the theory of Toda chain. We will have shown an equivalence of considered problems and also we give an interpretation of this criterion in terms of the John mapping. About links with some other problems of analysis see [58], [41], [43], [51].

Note that some results of present work have published already as short and incomplete fragments in papers [18], [19], [20].

Remark also that some explicit necessary and sufficient conditions of uniqueness breakdown of Dirichlet problem (and some others boundary value problems) solution for partial differential equations with constant coefficients have obtained (see e.g., [16] and [17]) in an arbitrary ellipse. Answers in that works were formulated

just in the form of condition (1.5) and prompted answers in our present investigations. In this paper we would like to open a way to examine ill-posed boundary value problems for partial differential equations in some more complicated domains than a circle along with a survey of another fields of mathematics about topics with equivalent contents.

The main goal of the present paper is thus to show numerous relations of the considered problem with remote branches of mathematics and mathematical physics.

The paper is organized as follows.

In Section 2 we describe a theory of uniqueness of Dirichlet problem and the John algorithm. In section 3 we apply this technique to the concrete choice of the biquadratic curve. In Section 4 we consider the solution of the Poncelet problem and show its relation with the John algorithm for the generic biquadratic curve. Section 5 is devoted to the Abel problem of reducing of elliptic integrals to elementary ones. We show how this problem can be formulated in terms of the Poncelet problem (or the John algorithm for the biquadratic curve). In Section 6 we consider relation of our problems with Ritt's problem of existence of periodic functions with a nontrivial multiplication property. Finally, in Section 7 we consider 3 problems in mathematical physics which are related with the Poncelet (or the John algorithm) problem: static solutions of the classical XY Heisenberg chain, elliptic solutions of the Toda chain and elliptic grids for biorthogonal rational functions in the theory of rational Padé interpolation .

2. Boundary value problems in a domain for the string equation

2.1. **Bibliographical remarks.** Investigations of ill-posed boundary value problems in bounded domains for partial differential equations go back to J.Hadamard [30] and then A.Huber [32] who for the first time noted nonuniqueness of the solution of the Dirichlet problem for the equation of string vibration (string equation) in a rectangular. Boundary value problems in bounded domains for nonelliptic partial differential equations were regularly studied on the whole in a parallelepiped, besides in domains with a general boundary questions of solution uniqueness of the Dirichlet problem for the hyperbolic equation of the second order on the plane (see reviews and results in [48] were usually studied. In work [14] D.Burgin and R.Duffin have examined the homogeneous Dirichlet problem for the equation $u_{tt} - u_{xx} = 0$ in a rectangular $\{0 \leq t \leq T; 0 \leq x \leq X\}$. It is shown that if the ratio T/X is irrational, uniqueness in space of continuously differentiated functions with summable second derivatives takes place. Theorems of existence of solutions in classical spaces are established, and smoothness of the solution is that more than smoothness of boundary function is big and than the number T/X is worse approximated by rational numbers. The Neumann problem is considered also. In works by B.Yo.Ptashnik and his pupils boundary value problems in a parallelepiped for a wide class of the differential equations and systems are investigated, see [48]. All these works (excepting the mentioned work [6]) are based on the methods, essentially using representation of domain as a topological product.

For nonrectangular domains the Dirichlet problem for the string equation was studied in connection with a number of the Denjoy-Poincare rotation (see, e.g., Z.Nitetsky's book [46]) of a homeomorphism of domain boundary, constructed on characteristics of equation (so-called an automorphism of characteristic billiards by

Fritz John [34]). A connection of properties of the Dirichlet problem with properties of this homeomorphism has used even in mentioned works by J.Hadamard and A.Huber. To the analysis this connection has undergone in F.John's work [34]. In works by R.A.Alexandrjan and his pupils investigations of this problem and, in particular, of this connection was continued ([3],[4], [47], [5]). The question on uniqueness of the solution of the Dirichlet problem in this ideology for domains that are convex with respect to characteristics families, should be transformed to a question on irrationality of number of rotation or, that is the same, to a question on presence a continuum set of finite orbits (cycles) of the discrete dynamic system generated by mentioned John homeomorphism. The same questions in connection with an asymptotic behavior of the solution of the Sobolev equation that describe surface oscillations of a fluid filling a body which is flying in atmosphere, were investigated by the siberian mathematicians T.I.Zelenjak, I.V.Fokin and others ([62], [24]). Researches of the string equation are included in well-known Yu.M.Berezansky's book [12] also, they give possibilities to build domains with angles in which the homogeneous Dirichlet problem is weakly solvable and well-posed concerning the right part and small moves of boundary of the domain, leaving angles in specified limits. Note more that the case of an ellipse was considered in works by A. Huber [32], Alexandrjan [4], V.I.Arnold [6], for small smoothness and for more general equations see the book [17].

2.2. **John condition.** For the problem (1.1), (1.2) in some general bounded domain Ω Fritz John [34] considered a remarkable transformation $T : C \to C$ of Jordan boundary into itself, allowing to do some conclusions about properties of the Dirichlet problem in Ω. Let us describe it.

Let Ω be arbitrary bounded domain which is convex with respect to characteristics of the equation (1.1), i.e. it has the boundary C intersected in at most two points by each straight line that is parallel to x- or y-axes. We start from arbitrary point M_1 on C and consider a vertical line passing through M_1. Generally, there are two points of intersection with the curve C: M_1 and some point M_2. We denote I_1 an involution which transform M_1 into M_2. Then, starting from M_2, we consider a horizontal line passing through M_2. Let M_3 be the second point of intersection with the curve C. Let I_2 be corresponding involution: $I_2M_2 = M_3$. We then repeat this process, applying step-by-step involutions I_1 and I_2. Denote $T = I_2I_1, T^{-1} = I_1I_2$. This transformation $T : C \to C$ gives us a discrete dynamical system on C, i.e. an action of group $\mathbb{Z}$ and each point $M \in C$ generates an orbit $\{T^nM | n \in \mathbb{Z}\}$. This orbit can be finite or denumerable set. The point M with finite orbit is called a periodic point and smallest n, for which $T^nM = M$, is called a period of the point M. In the paper [34] the uniqueness breakdown in the problem (1.1),(1.2) have studied in connection with topological properties of the mapping T for the case of even mapping T. The mapping T is called to be even or preserving an orientation if each positive oriented arc (P, Q) with points $P, Q \in C$ transforms into positive oriented arc (TP, TQ). F.John have proved several usefull assertions, among of which we extract the following one.

Sufficient condition of uniqueness. The homogeneous Dirichlet problem (1.1), (1.3) in the bounded domain has only a trivial solution in the space $C^2(\overline{\Omega})$ if the set of periodic points on C is finite or denumerable.

It has been selected four possible cases of dynamical system behavior:

I) all points are periodic (then their periods coincide);

II) there are periodic and nonperiodic points;

III) there are no periodic points and there is not a point, the orbit of which $\{..., T^{-1}P, P, TP, ..., T^nP, ...\}$ is dense in C;

IV) there are not periodic points and there is a point, the orbit of which $\{..., T^{-1}P, P, TP, ..., T^nP, ...\}$ is dense in C (transitive case).

In the work [34] it has shown that for a C^2-smooth curve the case III) is impossible. For the case II) it has proved that there is an arc D_0 on C such that each two arcs from $D_0, I_1D_0, TD_0, I_1TD_0, T^2D_0$, ..., $I_1T^{n-1}D_0, T^nD_0, ...$ have not any common point. Note that for the analytical boundary this is impossible because in this case T is a diffeomorphism. In the case IV) it has proved that the Denjoy-Poincaré rotation ξ ([46]) of the John mapping T is irrational and T is topologically equivalent to a turn of an unit circle about the angle $\pi\xi$ (i.e. there exists a homeomorphism h from C onto the unit circle S such that the mapping $hTh^{-1} : S \to S$ is a turn about the angle $\pi\xi$).

For the case when every point of C is periodic it have proved the all periods are coincided. But here we don't know anything about solution uniqueness, although R.A. Alexandrjan have shown in this case [3] that there is a generalized solution of the problem (1.1),(1.2) which can be built as a piecewise constant function.

We will assume that the domain has boundary C satisfying the condition:

The curve C is smooth and each characteristic line either doesn't intersect the curve C or touchs it at a point, named a vertex, or cuts C at two points. (2.1)

Note that above John's condition of convexity with respect of characteristic directions will be fulfilled under the condition (2.1) on $C = \partial\Omega$ in the case of a bounded domain Ω.

For the case of bounded domain with a biquadratic boundary we will see that above sufficient uniqueness condition is also necessary, moreover, it will be so even for cases when the curve C is unbounded but then we should change the setting of the problem. Namely, along with the usual setting of the uniqueness property:

the examined bounded domain is such that the homogeneous Dirichlet problem (1.1),(1.3) has only trivial solution in the space $C^2(\Omega) \cap C(\overline{\Omega})$, (2.2)

for cases when the curve C is unbounded we will examine the following modification of uniqueness property for the homogeneous Dirichlet problem :

the examined curve C is such that each analytic in real sense solution in $\mathbb{R}^2$ of the equation (1.1) with the property (1.3) is only zero solution. (2.3)

Note, the assumption "analytic" is introduced in order to allow unbounded curves C where there exist characteristic lines which do not intersect C and are between curve branches. Without this assumption on solutions for such curve and e.g. with an assumption of infinite smoothness for solutions one may build a simple example of smooth in $\mathbb{R}^2$ nontrivial solution of the equation (1.1) with the property (1.3). We will call a modified setting the problem to find an analytic in real sense solution in $\mathbb{R}^2$ of the equation (1.1) with the property (1.3).

Now we will give the John's proof of the sufficient condition of uniqueness in order to show that John's arguments are valid also for the problem 2.3.

Proposition 1. Under the condition (2.1) if the mapping T is transitive acted on C then 1) the uniqueness property (2.2) holds for the usual setting of the homogeneous

Dirichlet problem (1.1),(1.3) in a bounded domain and 2) the uniqueness property (2.3) holds for the modified setting in, possible, unbounded domain.

Proof. Let a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a nontrivial solution of the problem (1.1),(1.3) in a domain Ω with the condition (2.1). As it is well known, there exist two functions u_1, u_2 of the class C^2 depending on one variable such that $u(x,y) = u_1(x)+u_2(y)$ which we will write for any point $P \in C$ as $u(P) = u_1(P)+u_2(P)$. For the case of the property (2.3) consider a domain $\Omega \subset \mathbb{R}^2$ of points $P = (x_0, y_0) \in \mathbb{R}^2$ for which there exists a pair of different points of intersection $C \cap \{x = x_0\}$ of the curve with corresponding vertical line and also there exists a pair of different points of intersection $C \cap \{y = y_0\}$ of the curve with the horizontal characteristic line. Then for any point $P \in \Omega$ using definitions we easily obtain: $u_1(I_1P) = u_1(P)$, $u_2(I_2P) = u_2(P)$; $u_2(P) - u_2(TP) = u_2(P) - u_2(I_1P) = u(P) - u(I_1P) = 0$; $u_1(P)-u_1(T^{-1}P) = u_1(P)-u_1(I_2P) = u(P)-u(I_2P) = 0$. From what follows that equalities $u_1(P) = u_1(T^nP)$, $u_2(P) = u_2(T^nP)$ hold for each integer n. Continuity gives us that $u_1(P) = u_1(Q)$, $u_2(P) = u_2(Q)$ for any point Q from closure of the orbit of P. Because for the transitive acting the closing of the orbit of any point coincides with C then $u_1 \equiv const$ and $u_2 \equiv const$ and, therefore, $u \equiv 0$ in $\overline{\Omega}$. In the case of the property (2.3) the analyticity allows us to continue the zero solution onto the plane. □

Above we have noted that for any analytic boundary only two cases are admissible: periodic (I) and transitive (IV by John). Therefore we give the following setting of *the periodicity problem for John mapping* :

What curve from a given class of curves has the property:

$$\text{\textit{The John mapping }} T : C \to C \text{\textit{ has at least one periodic point.}} \tag{2.4}$$

Then as we have known already, the all points are periodic on our curve.

Along with above settings we will consider also a setting with complex John mapping. If $\tilde{C} \in \mathbb{C}^2$ is a biquadratic complex curve (i.e. one-dimensional complex variety (1.4)) then it is fulfilled the property of type 2.1:

$$\text{\textit{Almost each "vertical" line }} x = x_0 \text{\textit{ intersects the curve }} \tilde{C} \text{\textit{ at two different points and almost each "horizontal" line, too.}} \tag{2.5}$$

Let $\tilde{C} \in \mathbb{C}^2$ be an analytic curve with the same property. Then we can construct a John mapping T in the same way as in real case. And we can ask a similar question about *the periodicity problem for the complex John mapping* :

What curve from a given class of curves has the property:

$$\text{\textit{The complex John mapping }} T : \tilde{C} \to \tilde{C} \text{\textit{ has at least one periodic point.}} \tag{2.6}$$

Corresponding complex setting of the problem (1.1),(1.3) we will use is the following:

$$\text{\textit{To find two meromorphic functions }} f(x),\ g(y) \text{\textit{ of one complex variable such that }} f(x) + g(y) = 0 \text{\textit{ as soon as }} (x,y) \in \tilde{C} \subset \mathbb{C}^2. \tag{2.7}$$

Note that for the case of transitive action of the mapping T the proof of the proposition 1 one can easy transfer onto this complex case:

Proposition 2. For a biquadratic complex curve (1.4) if the complex John mapping T transitive acts on $\tilde{C}$ then the complex problem (2.7) has only trivial solution.

Below we will consider biquadratic curves C satisfying the property (2.1) and will give an explicit criterion distinguishing the cases of periodicity and transitivity and in the first case we will build an explicit nontrivial solution of the problem (1.1),(1.3) in sense of the settings 2.2 or 2.3 and as an intermediate the setting (2.7). In the case of transitivity each solution is proved to be zero.

2.3. **Boundary value problems and a moment problem.** In this subsection we are going to indicate an equivalence of properties of some boundary value problems, in particular of the Dirichlet and Neumann problems, and give a moment problem which is responsible for this properties. Thus we will obtain some problems that are equivalent to the problem (1.1),(1.3) in the setting (2.2). For details and generalizations see [16] or [17]. This equivalence is based on a connection condition of solution traces for the Cauchy problem.

Let us consider a hyperbolic equation in an arbitrary bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary

$$Lu = au''_{x_1x_1} + bu''_{x_1x_2} + cu''_{x_2x_2} = 0. \tag{2.8}$$

in the Sobolev space $H^m(\Omega), m \geq 3$.

Introduce a conormal vector ν_* and a derivative with respect to the conormal by means of an analog of the Green formula for the Laplace operator

$$\int_\Omega (Lu \cdot \overline{v} - u \cdot L\overline{v})\, dx = \int_{\partial\Omega} (u'_{\nu_*}\overline{v} - u\overline{v}'_*)\, ds.$$

One can count up that $\frac{\partial}{\partial \nu_*} = l(\nu)\frac{\partial}{\partial \nu} - \frac{1}{2k}[l(\nu(s))]'_s \cdot \frac{\partial}{\partial s}$, $l(\xi) = a\xi_1^2 + b\xi_1\xi_2 + c\xi_2^2$ is the symbol of the operator L, ν is an unit vector of normal, s is natural parameter on $\partial\Omega$, $k = \pm|\nu'_s|$ is the curvature, more exactly $\nu'_s = k\tau$ where $\tau = (-\nu_2, \nu_1)$ is the tangent vector.

And let's consider an overdetermined boundary value problem for the equation (2.8)

$$u'_s|_{\partial\Omega} = \gamma,\ u'_{\nu_*}|_{\partial\Omega} = \kappa. \tag{2.9}$$

that is otherwise written down the Cauchy problem $u|_{\partial\Omega} = \gamma_0,\ u'_\nu|_{\partial\Omega} = \kappa_0$.

We ask a question: what is a connection between the functions γ and κ if they are generated by a solution u of the equation (2.8)? In order to answer we need the following construction.

The equation (2.8) we write down also as

$$(\nabla \cdot a^1)(\nabla \cdot a^2)u = 0 \tag{2.10}$$

where $a^j = (a_1^j, a_2^j), j = 1, 2$ are unit real vectors. Let's enter vectors $\tilde{a}^1 = (-a_2^1, a_1^1)$, $\tilde{a}^2 = (-a_2^2, a_1^2)$, directing vectors of set of characteristic directions $\Lambda = \Lambda^1 \cup \Lambda^2$, $\Lambda^j = \left\{\lambda\tilde{a}^j | \lambda \in \mathbb{R}\right\}$, $j = 1, 2$, $\langle \tilde{a}^j, a^j \rangle = 0$. Let $\varphi_0 = \varphi_1 - \varphi_2$ where φ_j is any solution of equation $\tan\varphi_j = \lambda_j$, i.e. φ_j is an inclination angle of vector of the characteristic direction corresponding to the root $\lambda_{\mathbf{j}}$, φ_0 is angle between characteristics, and let $\Delta = \sin\varphi_0 = \det\|a^1\ a^2\|$, a_j are columns. Here and below a vector of a characteristic direction is understood as a vector $\nu \in \mathbb{C}^2$ which is a null of the symbol: $l(\nu) = 0$. The traces γ and κ of a solution u are linked by the following relations.

Statement 1. If a function $u \in H^m(\Omega)$, $m > 3$ is a solution of the problem (2.9) for the equation (2.8) then the functions $\gamma \in H^{m-3/2}(\partial\Omega)$, $\kappa \in H^{m-3/2}(\partial\Omega)$ from

(2.9) satisfy the conditions

$$\forall Q \in \mathbb{R}\,[z],\ \int_{\partial\Omega} \left[\kappa(s) + \frac{\Delta}{2}\gamma(s)\right] Q(x(s)\cdot \tilde a^1)\,ds = 0, \tag{2.11}$$

$$\forall Q \in \mathbb{R}\,[z],\ \int_{\partial\Omega} \left[\kappa(s) - \frac{\Delta}{2}\gamma(s)\right] Q(x(s)\cdot \tilde a^2)\,ds = 0, \tag{2.12}$$

where $x(s)$ is a moving point on $\partial\Omega$.

An inverse statement holds also:

Statement 2. If functions $\gamma \in H^{m-3/2}(\partial\Omega)$, $\kappa \in H^{m-3/2}(\partial\Omega), m > 3$ satisfy the conditions (2.11),(2.12) then there exists a solution $u \in H^{m-1-\epsilon}(\Omega)$ of the problem (2.9) for the equation (2.8) with each $\epsilon > 0$. For functions $\psi = u|_{\partial\Omega}$, $\chi = u'_\nu|_{\partial\Omega}$ we have $l(\nu)\chi = \kappa + [l(\nu)]'_\tau/2k$ and also $\psi = \int \gamma(s)ds + \text{const}$ (the Luzin's trigonometrical integral). In addition, function u is restored univalently up to an additive constant. The mapping:

$$H^m(\partial\Omega) \times H^m(\partial\Omega)/\{\text{const}\} \ni \{(\gamma,\kappa) \text{ with } (2.11),(2.12)\} \to u \in H^{m-1-\epsilon}(\Omega)$$

is continuous (for all $\epsilon > 0$).

Corollary 1. For each solution $u \in H^m(\Omega), m > 3$ of the equations (2.8) the following Zhukovsky's equality

$$\int_{\partial\Omega} \kappa ds = 0 \tag{2.13}$$

holds.

Proof. It follows from the condition (2.11) for $Q \equiv 1$ because $\int_{\partial\Omega} \gamma(s)ds = 0$. □

Consider following a moment problem:

$$\int_{\partial\Omega} \alpha(s)(x(s)\cdot \tilde a^j)^N ds = \mu_N^j;\ j = 1,2;\ N \in \mathbb{Z}_+,$$

where on two given vectors $\tilde a^j \in \mathbb{R}^2$ and on two sequences of numbers μ_N^j it is found the function α. Obviously, for the case when $\partial\Omega$ is the unit circle and vectors $\tilde a^j$, $j = 1,2$ are equal $\tilde a^1 = (1,i); \tilde a^2 = (1,-i)$ this moment problem turn on well-known trigonometric moment problem because then $(x(s)\cdot \tilde a^j)^N = \exp(\pm iN)$. Another way to the same is to write the Chebyshev polynomial T_N insteed of the power.

Among a lot of problems connected with above moment problem we will consider the problem of indeterminacy (uniqueness): for what curve $\partial\Omega$ and vectors $\tilde a^j$, $j = 1,2$ there exists a function α such that

$$\forall N \in \mathbb{Z}_+,\ j = 1,2,\ \int_{\partial\Omega} \alpha(s)(x(s)\cdot \tilde a^j)^N ds = 0. \tag{2.14}$$

The following fact takes place

Statement 3. Let $m \geq k \geq 3$ and let we have three sets of statements:

1_m) The homogeneous moment problem (2.14) has a nontrivial solution $\alpha \in H^{m-3/2}(\partial\Omega)$.

2_k) The Dirichlet problem $u|_{\partial\Omega} = 0$ for the equation (2.8) has a nontrivial solution $u \in H^k(\Omega)$.

3_k) The Neumann problem $u'_{\nu_*}|_{\partial\Omega} = 0$ for the equation (2.8) has a nonconstant solution $u \in H^k(\Omega)$.

Then $1_m) \Rightarrow 2_{m-q})$; $1_m) \Rightarrow 3_{m-q})$; $2_m) \Rightarrow 1_m)$; $3_m) \Rightarrow 1_m)$ with $q = 1+0$ (By definition, for bounded domain $H^{k+0}(\Omega) = \bigcup_{\epsilon>0} H^{k+\epsilon}(\Omega)$).

Proof. 1) $\Rightarrow$ 2). Using pair $\gamma = 0$, $\kappa = 2\alpha/\Delta$ with the help of the statement 1 we build the solution $u \in H^{m-q}(\partial\Omega)$.

2) $\Rightarrow$ 1). We put $\alpha = \kappa$ and apply the statement 2.

The implications 1) $\Rightarrow$ 3) and 3) $\Rightarrow$ 1) are similar. □

Note that instead of the Neumann problem in the statement 3 we can write the boundary condition of view

$$u_{\nu_*} = \lambda u_\gamma \tag{2.15}$$

with an arbitrary constant λ.

Note also that the case when the domain Ω is an ellipse has examined in the works by A. Huber [32], R. Alexandrjan [4], V.I. Arnold [6] and one of the authors [16] or [17]. An answer to the question about properties of such the Dirichlet problem (1.1),(1.3) is the following. Reduce by means of a linear transform our problem in an ellipse to the problem (1.3) in the unit disk for the equation (2.10). Find slope angles φ_1, φ_2 of characteristics and an angle $\varphi_0 = \varphi_1 - \varphi_2$ between them.

Statement 4. ([16], see also section 6) The problem (1.1),(1.3) has a nontrivial solution in the unit disk in a space $H^k(\Omega), k \geq 2$ if and only if

$$\varphi_0/\pi \in \mathbb{Q}. \tag{2.16}$$

If the condition (2.16) is fulfilled then there is a denumerable set of linear independent polynomial solutions of the problem (1.1),(1.3).

Thus, the existence of a nontrivial solution of the Dirichlet problem (1.1),(1.2) in a general bounded domain with smooth boundary is equivalent to the existence of a nonconstant solution of the boundary value problem (1.1),(2.15), in particular of the Neumann problem, and is equivalent to the existence of a nontrivial solution of the moment problem (2.14). Below we will give a criterion of nontrivial solvability each of these problems with the curve (1.4) in a view of the condition (2.16).

2.4. **Duality equation-domain and one more equivalent problem.** Let $\Omega \subset \mathbb{R}^n$ be a bounded semialgebraic domain given by means of the inequality $\Omega = \{x \in \mathbb{R}^n : P(x) > 0\}$ with a real polynomial P. Equation $P(x) = 0$ gives us the boundary $\partial\Omega$. We assume the boundary of domain Ω is nondegenerate: $\mid \nabla P \mid \neq 0$ on $\partial\Omega$. Consider the Dirichlet boundary value problem for the equation (2.8) of the order 2 with constant complex coefficients:

$$Lu = L(D_x)u(x) = 0, \quad u\mid_{\partial\Omega} = 0, \tag{2.17}$$

where $D_x = -i\partial/\partial x$. We understand a duality equation-domain as a correspondence the problem (2.17) and the equation:

$$P(-D_\xi)\{L(\xi)\, w(\xi)\} = 0, \tag{2.18}$$

given in the following statement:

Statement 5. For each nontrivial solution of the problem (2.17) from $C^2(\overline{\Omega})$ there exists unique nontrivial analytic in $\mathbf{C}^n$ solution of the equation (2.18) from some class Z of entire functions and conversely: for each nonzero solution $w \in Z$ of the

equation (2.18) there exists a nonzero solution $u \in C^2(\overline{\Omega})$ of the problem (2.17). The class Z is determined here as space of Fourier images of functions of the form $\theta_\Omega v$, where $v \in C^2(\mathbb{R}^n)$ and θ_Ω is the characteristic function of the domain Ω.

For a clearness we give here *a sketch of the proof.* We assume the problem has a nontrivial solution u in $C^2(\overline{\Omega})$, let $\tilde{u} \in C^2(\mathbb{R}^n)$ be any smooth continuation of u on the $\mathbb{R}^n$ then we multiple $\tilde{u}$ on characteristic function θ_Ω of domain Ω ($\theta_\Omega = 1$ in domain Ω and $\theta_\Omega = 0$ out Ω) and apply the operator L to the product $\theta_\Omega \tilde{u}$. Differentiating the product by Leibniz we obtain

$$L(\theta_\Omega \tilde{u}) = \theta_\Omega L(\tilde{u}) + L_1(u, \nabla u)\delta_{\partial\Omega} + L_2(u)(\delta_{\partial\Omega})'_\nu \tag{2.19}$$

where $\delta_{\partial\Omega}$ is measure supported on $\partial\Omega$: $< \delta_{\partial\Omega}, \phi >= \int_{\partial\Omega} \overline{\phi}(x) ds_x$, $L_2(u) = L(\nu)u$, ν is the external normal as earlier. The first term in (2.19) is equal to zero by means of equation. Take into account the boundary condition $u_{\partial\Omega} = 0$, obtain the last term will be transformed into term of view of the second term: $L_1(u, \nabla u)\, \delta_{\partial\Omega}$ as, for example, for the case of one variable $x\delta'(x) = -\delta$. Then we multiple the obtained equality on $P(x)$, the right-side part vanishes because e.g. for the case of one variable $x\delta(x) = 0$. We obtain an equation $P(x)L(D)(\theta_\Omega \tilde{u}) = 0$. Applying the Fourier transform $\mathcal{F}$ we obtain the equation (2.18) where $w = \mathcal{F}(\theta_\Omega \tilde{u})$. Necessity is proved. Sufficiency will be obtained by means of conversion of this proof. A full proof for a general case see in the works [17] or [15].

The term *'duality equation-domain'* means here an equivalence of the problems (2.17) and (2.18) that we will read here as:

$$L(D_x)u = 0, \quad u\,|_{P(x)=0} = 0, \tag{2.20}$$

$$P(-D_\xi)v = 0, \quad v\,|_{L(\xi)=0} = 0. \tag{2.21}$$

Now from statement 5 it follows

Proposition 3. If there exists a nontrivial solution of the problem (2.20) with the string equation (1.1), $L(\xi) = -\xi^1\xi^2$ where $\xi = (\xi^1, \xi^2)$ is a covector, in a plane bounded domain with the biquadratic curve (1.4) as a boundary $\partial\Omega$, then there exists also a nontrivial solution $v \in Z$ of the problem

$$\sum_{i,k=0}^{2} a_{ik} \frac{\partial^{i+j} v}{\partial \xi^i \partial \eta^k} = 0, \quad v|_{\xi=0} = 0,\ v|_{\eta=0} = 0 \tag{2.22}$$

and it is conversely.

The last boundary value problem for the equation of the fourth order has only two boundary condition insteed of fourth condition as it should be e.g. for the problem of a Goursat (or Darboux) type. Therefore it is no wonder that a nontrivial solution of the problem (2.22) exists there. But as we will show below for almost each curve (1.4) the problem (1.1), (1.2) has only trivial solution, therefore almost each problem (2.22) has only trivial solution as well. But this statement can seem astonishing namely because of the insufficiency of the data, in spite of the stipulation that the solution v belongs to the space Z.

We finished the statement of propositions on boundary value problems for general domains and now we should wait for section 6 in order to obtain explicit answers for the Dirichlet problem in domains with biquadratic boundary.

3. Generic biquadratic curve

3.1. **Parameterizations of generic biquadratic curve.** The complex curve (1.4) is remarkable by that this is the most general algebraic curve with the property that almost each vertical or horizontal line intersecting C intersects it in 2 points.

Let (1.4) be a generic nondegenerate real biquadratic curve. Assume that the parameters a_{ik} are chosen such that the real curve C bounding the domain satisfies the condition (2.1) of the subsection 2.2.

We will begin our study of the problem (1.1),(1.3) by an observation that the curve (1.4) is elliptic curve allowing an uniformization in terms of elliptic functions.

Indeed, rewrite the equality (1.4) in one of the form

$$A_2(x)y^2 + A_1(x)y + A_0(x) = 0 \tag{3.1}$$

or

$$B_2(y)x^2 + B_1(y)x + B_0(y) = 0, \tag{3.2}$$

where $A_i(x), B_i(y)$ are polynomials of degrees ≤ 2. We multiply the equation (3.1) by A_2, the equation (3.2) by B_2 and reduce these expressions to the forms

$$Y^2 - D_1(x) = 0 \quad \text{or} \quad X^2 - D_2(y) = 0, \tag{3.3}$$

where $Y = 2A_2(x)y+A_1(x),\ X = 2B_2(y)x+B_1(y);\ (x,y) \to (x,Y),\ (x,y) \to (X,y)$ are birational transformations and $D_1(x), D_2(y)$ are the discriminants of quadratic equations (3.1) and (3.2):

$$D_1(x) = A_1^2(x) - 4A_0(x)A_2(x),\ D_2(y) = B_1^2(y) - 4B_0(y)B_2(y).$$

In a general situation the discriminants $D_{1,2}$ are polynomials of 4-th or 3-d degree. Recall [60], that every curve of the kind

$$Y^2 = \pi_4(x) \tag{3.4}$$

with a generic 4-th degree polynomial $\pi_4(x)$, can be transformed to a canonical form

$$Y^2 = 4x^3 - g_2x - g_3 \tag{3.5}$$

which admits a standard parameterization

$$x = \wp(t),\ Y = \wp'(t) \tag{3.6}$$

in terms of the elliptic Weierstrass function $\wp(t)$ with primitive periods $2\omega_1,\ 2\omega_2$. The parameters g_2, g_3 are so-called (relative) invariants of the polynomial D_1. They are real for real π_4.

As $(x,Y) \to (X,y)$ is a birational transformation, the primitive periods of the curves (3.3) are coincide. The invariants g_2, g_3 may be found by periods. Hence we obtain the following important statement (which was mentioned by Halphen [31]):

Statement 6. Invariants g_2, g_3 of polynomials $D_1(x)$ and $D_2(x)$ are the same.

Thus, both curves (3.3) are elliptic ones [60] with appropriate primitive periods $2\omega_1, 2\omega_2$, the curve (1.4) is homeomorphic to the torus: $C \approx \mathbb{C}/(2\omega_1\mathbb{Z} \oplus 2\omega_2\mathbb{Z})$ and there are standard structures of a commutative group and an Abelian variety [29]. We deal with elliptic functions of the second order. Recall [60] that properties of a general elliptic function can be characterized by the number of poles (or that is equivalently, zeroes) in the fundamental parallelogram of periods. This number is called an order of the elliptic function (the order is taken with account of multiplicity

of poles). By the Liouville theorem, the simplest possible order is 2 [60]. E.g. the Weierstrass function $\wp(t)$ is of degree 2 because it has the only a pole of multiplicity 2 at the point $t = 0$ of the fundamental parallelogram.

The general elliptic function of the second order $\Phi(t)$ has two arbitrary poles p_1, p_2 and two arbitrary zeroes ζ_1, ζ_2 in the parallelogram of periods. The only condition is [60] $p_1 + p_2 - \zeta_1 - \zeta_2 = \Omega$, where $\Omega = 2m_1\omega_1 + 2m_2\omega_2$ is an arbitrary period. It can be easily showed that generic elliptic function of the second order with given periods $2\omega_1, 2\omega_2$ can be presented as

$$\Phi(t) = \frac{\alpha\wp(t-t_0)+\beta}{\gamma\wp(t-t_0)+\delta} \tag{3.7}$$

Thus, $\Phi(t)$ depends on 4 independent parameters, say $\alpha, \beta, \delta, t_0$.

There is another, sometimes more convenient, representation of the function $\Phi(t)$:

$$\Phi(t) = \kappa\,\frac{\sigma(t-e_1)\sigma(t-e_2)}{\sigma(t-d_1)\sigma(t-d_2)}, \tag{3.8}$$

where κ is a constant and parameters e_1, e_2, d_1, d_2 are related as

$$e_1 + e_2 = d_1 + d_2. \tag{3.9}$$

The form (3.8) is obtained from standard representation of the arbitrary elliptic function in terms of the Weierstrass sigma-functions [60]. In this case the points e_1, e_2 and d_1, d_2 coincide with zeros and poles of the function $\Phi(t)$ and the condition (3.9) is equivalent to a balance condition between poles and zeros of the generic elliptic function.

Note that apart from $\wp(t)$ there are another special cases of functions of second order, e.g. the Jacobi elliptic functions $sn(t;k), cn(t;k), dn(t;k)$ [60] that we will use below.

What is an uniformization for the biquadratic curve? The answer is given by following

Theorem 1. The generic complex biquadratic curve (1.4) can be parameterized by a pair of elliptic functions of the second order with the same periods:

$$x(t) = \Phi_1(t), \quad y(t) = \Phi_2(t) \tag{3.10}$$

Conversely, any two elliptic functions $x = \Phi_1(t), y = \Phi_2(t)$ of the second order with the same periods satisfy an equation (1.4).

This theorem is proved essentially e.g. in the famous Halphen monograph [31] on elliptic functions. We give a proof based on some main Halphen's ideas here.

Proof. From (1.4) it follows the Euler differential equation ([33])

$$\frac{dx}{\sqrt{D_1(x)}} = \pm\frac{dy}{\sqrt{D_2(y)}} \tag{3.11}$$

because $dy/dx = -F_x/F_y = -X/Y$ (see (3.3)).

As we already saw, the polynomials $D_1(x)$ and $D_2(y)$ have the same invariants g_2, g_3. Hence they both can be reduced to the same canonical Weierstrass form (see, e.g. [1], s.34) by means of a pair of the Möbius transforms

$$\tilde{x} = \frac{\mu_1 x + \nu_1}{\xi_1 x + \eta_1}, \quad \tilde{y} = \frac{\mu_2 y + \nu_2}{\xi_2 y + \eta_2}, \quad \mu_i\eta_i - \nu_i\xi_i = 1,\ i = 1,2 \tag{3.12}$$

with some complex parameters $\mu_1, \dots \eta_2$. Hence the equation (3.11) becomes

$$\frac{d\tilde{x}}{\sqrt{4\tilde{x}^3 - g_2\tilde{x} - g_3}} = \frac{d\tilde{y}}{\sqrt{4\tilde{y}^3 - g_2\tilde{y} - g_3}} \tag{3.13}$$

because $d\tilde{y}/d\tilde{x} = -\tilde{X}/\tilde{Y}$ as above. But the equation (3.13) means that for appropriate periods $2\omega_1, 2\omega_1$

$$\tilde{x} = \wp(u), \quad \tilde{y} = \wp(u + u_0), \tag{3.14}$$

where $\wp(u) = \wp(u; \omega_1, \omega_2)$, u is an uniformization parameter and u_0 is a complex constant. Now we can return to initial variables x, y by means of inverse Möbius transforms to get finally

$$x = \frac{\alpha_1\wp(u) + \beta_1}{\gamma_1\wp(u) + \delta_1}; \quad y = \frac{\alpha_2\wp(u + u_0) + \beta_2}{\gamma_2\wp(u + u_0) + \delta_2}, \tag{3.15}$$

with some constants $\alpha_i, \dots, \delta_i$: $\alpha_i\delta_i - \beta_i\gamma_i = 1,\ i = 1, 2$. Then we apply formula (3.7) which says that we indeed obtained a pair of elliptic functions of the second order.

The inverse statement of the theorem follows from a general theorem that any two elliptic functions $x(t)$ and $y(t)$ with the same periods satisfy an algebraic equation $F(x(t), y(t)) = 0$. The degrees of the polynomial $F(x, y)$ with respect to variables x and y are determined by orders of corresponding elliptic functions. If both functions have the order 2 then the polynomial $F(x, y)$ has the degree at most 2 with respect to each variable (this statement is contained in [60] as a problem for the reader). □

Thus, we proved the theorem 1. Moreover, simultaneously we have proved the following

Proposition 4. There exist complex linear-fractional transforms (3.12) such that transformed generic curve (1.4) having the same form can be parameterized by only the Weierstrass function $\wp(u)$ as in (3.14).

Considering the formulae (3.10) we can write down our parameterization in terms of the Weierstrass sigma-function:

$$x(t) = \kappa_1 \frac{\sigma(t - e_1)\sigma(t - e_2)}{\sigma(t - d_1)\sigma(t - d_2)}, \quad y(t) = \kappa_2 \frac{\sigma(t - e_3)\sigma(t - e_4)}{\sigma(t - d_3)\sigma(t - d_4)} \tag{3.16}$$

with two restrictions $e_1 + e_2 = d_1 + d_2$ and $e_3 + e_4 = d_3 + d_4$. If now we make a shift $t \to t - t_0$ of the uniforming parameter t then we can choose t_0 such that, say, $e_2 = -e_1, d_2 = -d_1$. This means

Proposition 5. In formulae (3.10) the function $\Phi_1(t)$ can be chosen even: $\Phi_1(-t) = \Phi_1(t)$ by means of choice a shift $t \to t - t_0$.

From our considerations it follows an important corollary (which was mentioned by Halphen [31] as well):

Proposition 6. Consider the differential equation (3.11). Let $D_1(x)$ and $D_2(y)$ be polynomials of degree 4 or 3 with the same invariants g_2, g_3. And let $(x(t), y(t))$ be a solution of this equation (parameterized e.g. by an initial condition). Then $x(t), y(t)$ satisfy a biquadratic equation of the form (1.4).

3.2. **Biquadratic foliation and singular points.** There is an interesting mechanical interpretation of last results. Assume that we have a dynamical Hamiltonian system for two canonical variables $x(t), y(t)$ satisfying a system of equations

$$\dot{x} = \frac{\partial H(x,y)}{\partial y}, \quad \dot{y} = -\frac{\partial H(x,y)}{\partial x}, \tag{3.17}$$

where $H(x,y)$ is a Hamilton function of the system. Obviously $H(x,y)$ is an integral of the system (3.17), i.e. $(H(x,y))^{\cdot} = 0$. Choose the Hamiltonian as the biquadratic function (1.4): $H(x,y) = F(x,y)$. We have $F(x,y) = c$ with some constant c depending on initial conditions for x, y. This constant can be incorporated to the coefficient a_{00}, so we can write down $\tilde{F}(x,y) = 0$, where $\tilde{F}(x,y) = F(x,y) - c$ is again a biquadratic curve (note that for $\tilde{F}(x,y)$ the coefficients $A_2(x), A_1(x), B_2(y), B_1(x)$ remain the same whereas the coefficients $\tilde{A}_0(y)$ and $\tilde{B}_0(y)$ differ from initial ones by a constant). Then

$$\frac{\partial H(x,y)}{\partial y} = 2A_2(x)y + A_1(x) = \pm\sqrt{\tilde{D}_1(x)},$$

where $\tilde{D}_1(x) = A_1^2(x) - 4A_2(x)\tilde{A}_0(x)$ (because y can be excluded as a root of quadratic equation $A_2(x)y^2 + A_1(x)y + \tilde{A}_0(x) = 0$). Quite analogously

$$\frac{\partial H(x,y)}{\partial x} = 2B_2(y)x + B_1(y) = \pm\sqrt{\tilde{D}_2(y)}.$$

We thus see that for any fixed Hamiltonian level $H = c$ the variables $x(t), y(t)$ satisfy the Euler equation (3.11) where polynomials $\tilde{D}_1(x)$, $\tilde{D}_2(y)$ have the same invariants $\tilde{g}_2, \tilde{g}_3$. Note, that in this case the invariants $\tilde{g}_2, \tilde{g}_3$ (and hence the periods $2\omega_1, 2\omega_2$) will depend on the value of integral c. We thus obtain a whole one-parameter family of biquadratic curves $F(x,y) = c$ and corresponding elliptic functions $x(t), y(t)$ of the second order that are trajectories of this dynamical system.

Rewrite discriminants $\tilde{D}_1(x), \tilde{D}_2(y)$ in factorized forms

$$\tilde{D}_1(x) = q_1 \prod_{i=1}^{4}(x - x_i), \quad \tilde{D}_2(y) = q_2 \prod_{i=1}^{4}(y - y_i),$$

where q_1, q_2 are leading coefficients of the discriminants and $x_i, y_i, i = 1,2,3,4$ are their roots (for simplicity we assume that both discriminants have degree 4). In general, roots x_i, y_i will depend on the parameter c. What is mechanical meaning of points x_i, y_i? From equations of motion it is seen that

$$\dot{x} = \pm\sqrt{\tilde{D}_1(x)}, \quad \dot{y} = \pm\sqrt{\tilde{D}_2(y)}$$

Thus x_i and y_i are *stable points*: $\dot{x}_i = \dot{y}_i = 0$. We should demand that the points (x_i, y_k) belong to our biquadratic curve $\tilde{F}(x,y) = 0$. This leads to conditions that points (x_i, y_k) satisfy the conditions

$$\tilde{F}(x,y) = 0, \quad \partial_x \tilde{F}(x,y) = 0, \quad \partial_y \tilde{F}(x,y) = 0 \tag{3.18}$$

which in turn means that points (x_i, y_k) are singular points of the biquadratic curve $\tilde{F}(x,y) = 0$. As we saw, in generic situation this curve is elliptic and hence has a genus 1 (genus $C = C_{n-1}^2 - d$, n is degree and d is a number of double points of C). The degree of this curve is 4. Assume that the curve is irreducible. Then such curve cannot have more than 3 singular points in complex projective plane. The

latter is defined by the coordinates (s_0, s_1, s_2) such that $x = s_1/s_0, y = s_2/s_0$. In these coordinates we have equation of our curve

$$a_{22}s_1^2s_2^2 + s_0s_1s_2(a_{12}s_1 + a_{21}s_2) + s_0^2(a_{20}s_1^2 + a_{02}s_2^2 + a_{11}s_1s_2) + \\ s_0^3(a_{10}s_1 + a_{01}s_2) + s_0^4\tilde{a}_{00} = 0. \tag{3.19}$$

Elementary considerations show that two points $(0, 1, 0)$ and $(0, 0, 1)$ of the projective plane are singular for any values of parameters a_{ik}. Thus, only one singular point can exist in each finite part of the plane. In turn, this means that at least two roots, say x_1, x_2 of the discriminant $\tilde{D}_1(x)$ should coincide: $x_1 = x_2$. The same condition holds for the discriminant $\tilde{D}_2(y)$, i.e. $y_1 = y_2$ because invariants g_2, g_3 of the discriminants $\tilde{D}_1(x)$ and $\tilde{D}_2(y)$ are the same. Then it is elementary verified that the point (x_1, y_1) will be indeed a singular point of the biquadratic curve. In principle, the second singular point can occur. But in this case the genus of the curve will be -1 meaning that the curve is reducible.

We thus have the following

Proposition 7. The irreducible biquadratic curve $\tilde{F}(x, y) = 0$ has the a singular point in a finite part of the (complex) plane if and only if the discriminant $\tilde{D}_1(x)$ (and hence $\tilde{D}_2(y)$ as well) has a multiple zero x_1 (correspondingly y_1). In this case the point (x_1, y_1) is singular and unique.

Note this proposition can be reformulated in an equivalent form. Indeed, the polynomial $D_1(x)$ has a multiple zero if and only if its discriminant is zero. Thus, in order to find all singular points of the curve we should first calculate the discriminant $D_1(x)$ of the biquadratic curve $F(x, y) = 0$ and then calculate the discriminant $\Delta(D_1(x))$ from the polynomial $D_1(x)$ ("discriminant from the discriminant"). We have obviously coincidence of the two discriminants

$$\Delta(D_1(x)) = \Delta(D_2(y))$$

because the invariants g_2, g_3 of the polynomials $D_1(x)$ and $D_2(y)$ are the same (statement 6). Condition $\Delta(D_1(x)) = 0$ (or, equivalently, $\Delta(D_2(y)) = 0$) gives us some nonlinear equations for the coefficients a_{ik}. Under such condition the biquadratic curve $F(x, y) = 0$ has a genus < 1, i.e. it is either irreducible and has the only singular point in a finite part of the projective plane, or it is reducible: $F(x, y) = \tau_1(x, y)\tau_2(x, y)$, where $\tau_1(x, y), \tau_2(x, y)$ are two polynomials linear with respect to each argument x, y (but $\tau_k(x, y)$ are not in general linear functions, they may contain the terms like xy). We have obtained

Proposition 8. The condition $\Delta(D_1(x))$ $(= \Delta(D_2(y))) \neq 0$ is necessary and sufficient for the equality: genus $C = 1$.

3.3. **Case of generic symmetric curve.** Above we considered generic case when our biquadratic curve $F(x, y) = 0$ is non-symmetric.

Assume now that our curve is symmetric, i.e. $F(x, y) = F(y, x)$. Equivalently, this means the coefficient matrix a_{ik} in (1.4) is symmetric: $a_{ik} = a_{ki}$. In this case, obviously, both discriminants coincide $D_1(x) = D_2(x)$. From the Euler differential equation (3.11) we obtain that a parameterization can be given by formulae

$$x(u) = \Phi(u), \quad y(u) = \Phi(u + u_0), \tag{3.20}$$

where u_0 is a constant and $\Phi(u)$ is an even function of the second order, i.e. $\Phi(t)$ can be presented in the form

$$\Phi(u) = \frac{\alpha\wp(u) + \beta}{\gamma\wp(u) + \delta} \tag{3.21}$$

Thus, in the symmetric case a parameterization is provided by some even elliptic function of the second order. Vice versa, any pair $(x, y) = (\Phi(u), \Phi(u + u_0))$ with arbitrary u_0 generates a symmetric biquadratic curve by formulas (3.20) because the point $(y, x) = (\Phi(u + u_o), \Phi(u)) = (\Phi(\tilde{u}), \Phi(\tilde{u} + u_0))$, $\tilde{u} = -u - u_0$ belongs C also. Note that the last statement was attributed to Euler in the work [59].

We obtain the following

Proposition 9. The generic complex symmetric biquadratic curve (1.4) can be parameterized by an even elliptic function of the second order and a shift as in (3.20). Conversely, for each elliptic function $\Phi(u)$ (even or not) of the second order and any shift u_0 the variables x, y from (3.20) satisfy the equation (1.4) with a symmetric matrix A.

This implies the important

Proposition 10. The generic complex biquadratic curve (1.4) can be transformed to symmetric one by means of a linear-fractional complex changes of variables (3.12). If the initial curve is real then it can be transformed to real symmetrical curve also (although corresponding transformation can be with complex coefficients).

Proof. As we have shown in the theorem 1 (see proposition 4), generic non-symmetric curve can be transformed to a curve described as (3.14) by linear-fractional complex changes (3.14) that means the curve-image is symmetric one in virtue of the last proposition. In the real case we note that as it is well-known, the invariants g_2, g_3 of real polynomial $D_1(x)$ are real, so that the differential equation (3.13) is real and there exists its real solution $(x(u), y(u))$ that can be extended onto complex domain and therefore satisfies the symmetric equation (1.4). The last equation has irreducible polynomial which must be real if $a_{22} = 1$ because we always can choise real parameters $u_1, ...u_8$ such that vectors V_i composed of components $x^k(u_i)y^l(u_i),\ 0 \le k, l \le 2, 0 \le k + l < 4$ will be linear independent, then the coefficients of our polynomial satisfy a linear system of 8 linear equations with real coefficients $(V_i)_j$ and the real right-side parts $-x^2(u_i)y^2(u_i)$. □

Note that another proof of the last fact is there in the work [35], see below statement 7.

Let us now consider the question how restore the polynomial F if we know the discriminants D_1, D_2. In symmetric case T. Stieltjes [55] proposed a nice explicit formula for the polynomials $F(x, y)$ by means of a solution of the differential equation (3.11). Assume that

$$D_1(x) = D_2(x) = b_0x^4 + 4b_1x^3 + 6b_2x^2 + 4b_3x + b_4.$$

Then $F(x, y)$ can be given by the determinant:

$$F(x,y) = \begin{vmatrix} 0 & 1 & -(x+y)/2 & xy \\ 1 & b_0 & b_1 & b_2 - 2C \\ -(x+y)/2 & b_1 & b_2 + C & b_3 \\ xy & b_2 - 2C & b_3 & b_4 \end{vmatrix} \tag{3.22}$$

The function $F(x,y)$ is defined up to an arbitrary non zero number factor. It is assumed that the curve is nondegenerated, i.e. its genus is 1. This is possible if and only if the determinant

$$\Delta = \begin{vmatrix} b_0 & b_1 & b_2 - 2C \\ b_1 & b_2 + C & b_3 \\ b_2 - 2C & b_3 & b_4 \end{vmatrix}$$

is nonzero. C is an integration (arbitrary) constant ([55]). The Stieltjes formula is useful in the problem of reducing of the arbitrary symmetric biquadratic curve to some standard forms, as we will see below.

3.4. **Canonical forms of biquadratic curve.** The general curve (1.4) contains 8 free parameters. It is naturally to transform this curve to the form containing the smallest possible numbers of parameters.

First, note that under arbitrary projective transformations

$$x \to \frac{\xi_1 x + \eta_1}{\mu_1 x + \nu_1}, \quad x \to \frac{\xi_2 x + \eta_2}{\mu_2 x + \nu_2} \tag{3.23}$$

with complex parameters we obtain similar equation (1.4) but with transformed parameters a_{ik}. This idea has already exploited in the proof the theorem 2. Every projective transformation (3.23) contains 3 independent parameters, hence it is possible to reduce the total number of independent parameters a_{ik} to $8-6=2$. As our curve is an elliptic one these free parameters are only invariants g_2, g_3 under linear-fractional transformations of variables x and y separately. More explicitly, if we consider the projectivisation $\sum\limits_{i_1,i_2,j_1,j_2=1}^{2} a_{i_1 i_2 j_1 j_2} x_1^{i_1} x_2^{i_2} y_1^{j_1} y_2^{j_2},\ x = x_1, y = y_1$ of initial curve (1.4) and its projective transformations of variables x and y separately that g_2, g_3 are only invariants (of respective wights 4 and 6) but for the transformations of the group $SL(2,\mathbb{C})$ these are absolute invariants.

Above we reduce the general case to the symmetric one so that we restrict ourselves with considering a symmetric curve $F(x,y) = F(y,x)$. So we would like to find some canonical forms of the curve containing only 2 parameters.

There are two obvious canonical forms which can be obtained for the curve $F(x,y) = 0$. These two forms correspond to two canonical forms of elliptic integrals in the Euler differential equation (3.11).

(I) The first one can be obtained if one reduces polynomial $D_1(x) = D_2(x)$ to the canonical cubic Weierstrass form:

$$D_1(x) = 4x^3 - g_2 x - g_3.$$

Such form can be always achieved by an appropriate projective transformation (with possible complex coefficients). Then from the Stieltjes formula (3.22) we obtain the expression (see also [31])

$$F(x,y) = (xy + (x+y)w + g_2/4)^2 - (x+y+w)(4xyw - g_3) = 0, \tag{3.24}$$

where $w = C$ is an arbitrary parameter. There is a parameterization of this curve in terms of the Weierstrass elliptic function

$$x(u) = \wp(u),\ y(u) = \wp(u + u_0)$$

and $w = \wp(u_0)$, where u_0 is an arbitrary constant. It is easy to calculate the discriminant:

$$D_1(x) = (4w^3 - g_2 w - g_3)(4x^3 - g_2 x - g_3)$$

If the parameter w is such that $4w^3 - g_2 w - g_3 = 0$ then $D_1(x) \equiv 0$ and in this case the curve $F(x, y)$ is reducible: $F(x, y) = \rho^2(x, y)$, where $\rho(x, y)$ is a polynomial of degree 1 with respect to each variable x, y. If $4w^3 - g_2 w - g_3 \neq 0$ then the curve is irreducible. The singular points in a finite part of the complex projective plane appear only if $g_2^3 = 27g_3^2$. This condition means that the polynomial $4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$ has a double root, say $e_1 = e_2$. In this case we can put $g_2 = 3\tau^2$, $g_3 = \tau^3$, where τ is some parameter. Then it is elementary verified that the point $x = y = -\tau/2$ will be the only finite singular point of the curve $F(x, y) = 0$. If $g_2^3 - 27g_3^2 \neq 0$ then there are no singular points in the finite part of the projective plane and the curve (3.24) is irreducible and has a genus 1.

The curve (3.24) contains 3 parameters w, g_2, g_3. Assume that $4w^3 - g_2 w - g_3 = W \neq 0$. In this case the curve is irreducible. By the linear transformation of arguments $x \to \alpha x + w$, $y \to \alpha y + w$, where $\alpha^3 = 1/W$ we can eliminate terms $x^2 y, xy^2$ and $x^2 + y^2$. The curve then is reduced to the form

$$F(x, y) = x^2 y^2 - x - y + Axy + B = 0 \tag{3.25}$$

containing only 2 independent parameters A, B. The curve is elliptic non-singular (i.e. having genus 1) if condition

$$\Delta(D_1) = B(A^2 - 4B)^2 + A(36B - A^2) - 27 \neq 0 \tag{3.26}$$

holds.

Thus we proved the following

Proposition 11. The generic complex biquadratic curve (1.4) can be transformed to canonical form (3.25) by means of a linear-fractional complex changes of variables (3.12). If the initial curve is real then it can be transformed to real form also (although corresponding transformation can be with complex coefficients).

As we saw, any irreducible biquadratic curve can be transformed to this form. However, here we need general projective transformations (3.23) with complex parameters. This will be so even in the case when all parameters a_{ik} of the biquadratic curve are real.

(II) In beginning let remember well-known Legendre transformation. Assume that an elliptic curve has already the form $Y^2 = \pi_4(x)$ (see (3.3)), where $\pi_4(x)$ is an arbitrary polynomial of the 4-th degree with real coefficients. Using only a linear-fractional transformation $x = \Gamma(\widetilde{x})$ with real coefficients it is possible to reduce this curve to the form (with a new Y) where $\pi_4(x)$ has only even degrees of x: $\pi_4(x) = \alpha x^4 + \beta x^2 + \gamma$ with some real coefficients α, β, γ (see, e.g. [23]).

Assume now that our biquadratic curve is symmetric and has only real coefficients a_{ik}. Then we can always transform this curve to a form without terms of odd degrees, i.e. $a_{21} = a_{12} = a_{10} = a_{01} = 0$. Indeed, taking $x = y$ in expression (1.4) for F we obtain the polynomial $F(x, x)$ of fourth degree and apply to it the Legendre change $x = \Gamma(\widetilde{x})$ in a generic case that reduces the equation $F(x, x) = 0$ to the view $\alpha\widetilde{x}^4 + \beta\widetilde{x}^2 + \gamma = 0$. Therefore the change with the same Γ

$$x = \Gamma(\widetilde{x}), \;\; y = \Gamma(\widetilde{y}) \tag{3.27}$$

applied to (1.4) gives us an equation with desired polynomial (observation from [35]).

We then obtain so called the Euler-Baxter biquadratic [10], [58]:

$$F(x,y) = x^2y^2 + a(x^2+y^2) + 2bxy + c = 0, \tag{3.28}$$

where a, b, c are remaining real parameters if the initial curve was real and, obvious, will be complex if the initial curve was complex. The curve (3.28) plays the crucial role in deriving addition theorem for the elliptic function $sn(t)$ [1]. It appeared also in Baxter's approach to 8-vertex model in statistical mechanics [10].

We first analyze possible finite singular points of the curve (3.28). The discriminant of the curve (3.28) is equal to

$$D_1(x) = -ax^4 - (a^2-b^2+c)x^2 - ac = -a(x^4 - \tilde{b}x^2 + c);\ \tilde{b} = \frac{b^2-a^2-c}{a}, \tag{3.29}$$

$D_1(x) = D_2(x)$ and by proposition 7 the curve (3.28) has a finite singular point under the condition

$$\Delta(D_1(x)) = 16a^2c\left((a^2-b^2+c)^2 - 4a^2c\right)^2 = 0.$$

If $c = 0$ then we have an obvious singular point $x = y = 0$. This singular point will be isolated for all values of the parameters a, b excepting the case $a \pm b = 0$. If $a \pm b = 0$ then the curve $F = 0$ becomes reducible. In any case if $c = 0$ the genus of the curve (3.28) is less then 1 (i.e. 0 for generic case and -1 for exceptional case corresponding to a reducible curve). Thus in this case the curve is not elliptic and can parameterized by rational functions. Here the change $x \to \kappa/x$, $y \to \kappa/y$ gives us a form $F = x^2 + y^2 + axy \pm 1$.

Using then scaling transformations $x \to \kappa x$, $y \to \kappa y$ in real case we can reduce the coefficients $c \neq 0$ to ± 1 depending on sign of this coefficient.

Thus, in real symmetric case we have simplest forms:
(i) $F = x^2y^2 + a(x^2+y^2) + 2bxy + 1$;
(ii) $F = x^2y^2 + a(x^2+y^2) + 2bxy - 1$.
In (ii) one can consider $a > 0$ by virtue of substitution $x \to 1/x, y \to 1/y$.

The canonical biquadratic curve (1.4) with F from (i) or (ii) we will call the Euler-Baxter curve (as (3.28)).

Proposition 12. The generic complex biquadratic curve (1.4) can be transformed to canonical form (i) by means of a linear-fractional complex changes of variables (3.12). If the initial curve is real then it can be transformed to real form (i) or (ii) (although corresponding transformation can be with complex coefficients). By means of only some real linear-fractional transformation (3.12) any generic real symmetric biquadratic curve (1.4) can be reduced to one of two form: (i), (ii).

Remark that in the paper [35] it is proved the following

Statement 7. By means of only some real birational transformation any generic real elliptic biquadratic curve (1.4) can be reduced to one of three form: (i), (ii) or to the form
(iii) $F = x^2y^2 + a(x^2-y^2) + 2bxy - 1$.

Note that the discriminants of the curve (iii) are equal to

$$D_1(x) = -a(x^4 - \hat{b}x^2 + 1);\ D_2(y) = a(y^4 + \hat{b}y^2 + 1);\ \hat{b} = \frac{b^2+a^2+1}{a}. \tag{3.30}$$

In the complex case we can reduce the coefficients $c \neq 0$ to 1 and leave only the case (i) among nondegenerate cases. Here following [10] we can find a parameterization of the curve (3.28) in terms of elliptic Jacobi function sn:

Proposition 13. The curve (3.28) can be parameterized by the formulae

$$x = \sqrt{k}\, sn(t; k), \quad y = \sqrt{k}\, sn(t \pm \eta; k), \tag{3.31}$$

with any sigh $\pm$, where parameters k, η are determined by relations

$$k + k^{-1} = (b^2 - a^2 - 1)/a, \quad 1 + k\, a\, sn^2(\eta; k) = 0. \tag{3.32}$$

Note that we can replace $t = \tilde{t} + K$ in order to deal with an even function. Note that the mapping (3.31) is an analytic diffeomorphism from fundamental parallelogram, factored by a standard way into torus, onto $\tilde{C}$.

Real parameterizations of curves (i), (ii) and (iii) are brought in [35] and [36].

Now let's remember expressions (3.3) writing as

$$F_x^2(x_1, y_1) = D_2(y_1) = B_1^2(y_1) - 4B_0(y_1)B_2(y_1). \tag{3.33}$$

We deduce y-extreme point (x_1, y_1) gives us $0 = y'(x_1) = F_x/F_y(x_1, y_1)$; $0 = F_x(x_1, y_1) = 2B_2(y_1)x_1 + B_1(y_1)$. If $B_2(y_1) = 0$ then $x_1 = -B_0(y_1)/B_1(y_1)$ from (3.2), otherwise $F_x^2(x_1, y_1) = D_2(y_1) = 0$, $x_1 = -B_1(y_1)/2B_2(y_1)$. Thus, for any y-vertex (x_1, y_1) the number y_1 is a root of the polynomial $D_2(y_1)$. Conversely, from the equalities (3.33) and $y'F_y = -F_x$ we see that for each such root y_1 of D_2 the point (x_1, y_1) is either y-vertex (extreme point in y-direction) or a singular point. But as we have seen in subsection 3.2 a singular point can be only if a root of D_2 is multiple, i.e. $\Delta = g_2^3 - 27g_3^2 = 0$; this is an exceptional case. Therefore it is proved the following

Proposition 14. For generic case the roots of $D_2(y_1)$ and only they are y-coordinates of y-vertexes of the curve. It's analogous, for generic case the roots of $D_1(x_1)$ and only they are x-coordinates of x-vertexes.

Now we can give geometrical an interpretation of cases (i), (ii), (iii), a proof of which can be easy obtained by analysis of formulae (3.29) and (3.30).

Proposition 15. 0). The real curve in cases (i), (ii) and (iii) disappears only in the case (i), $\tilde{b} < 2$, $a > 0$ that is equivalent to $b^2 < (a+1)^2$, $a > 0$.

1). In case (i) of nonvanishing real curve the equation $D_1(x) = 0$ (and the equation $D_2(y) = 0$) has no real root only in the case $\tilde{b} < 2$, $a < 0$ (there is no vertex).

2). In case (i) the equation $D_1(x) = 0$ (and the equation $D_2(y) = 0$) has four real roots under $\tilde{b} > 2$ (there are 4 x-vertices and 4 y-vertices).

3). In the case (ii) the equation $D_1(x) = 0$ (and the equation $D_2(y) = 0$) has two real roots (there are 2 x-vertices and 2 y-vertices).

4). In the case (iii) for $a > 0$ ($\hat{b} > 2$) the equation $D_1(x) = 0$ has 4 real roots and the equation $D_2(y) = 0$ has no real root (there are 4 x-vertices and no y-vertex).

5). In the case (iii) for $a < 0$ ($\hat{b} < -2$) the equation $D_1(x) = 0$ has no real root and the equation $D_2(y) = 0$ has four real roots (there are 4 y-vertices and no x-vertex).

If we will allow values $x = \infty$, $y = \infty$ i.e. consider our real curve on a thorus $S^1 \times S^1$ then the number of x-vertexes and the number of y-vertexes will not be changed by Möbius transformations (3.23), therefore we obtain

Proposition 16. Linear-fractional changes (3.23) don't remove the curve from its class 0)-5) of the proposition 15.

3.5. **John mapping of biquadratic curve and periodicity.** At first, we show how complex John algorithm works for the curve C. Consider a case of some more general curve.

Let $\tilde{C} \subset \mathbb{C}^2$ be a complex curve described parametrically as

$$x(t) = \phi(t), \quad y(t) = \phi(t+\varepsilon), \quad t \in \mathbb{C} \tag{3.34}$$

where ε is a nonzero complex parameter. Assume that $\phi(z)$ is an even periodic meromorphic function, i.e. $\phi(-z) = \phi(z)$, $\phi(z+T) = \phi(z)$ with some (complex) constant T. And let $C = \tilde{C} \cap \mathbb{R}^2$ be a real curve given by means of a contraction of the functions x, y on some set $S \in \mathbb{C}$.

We will assume that the curve $\tilde{C}$ satisfies the condition:

The curve $\tilde{C}$ is nondegenerate and each straight line of the form $x = x_0$
or $y = y_0$ intersects the curve $\tilde{C}$ at no more than two points (3.35)

Replace $t \to -t$. Then, due to the fact that $\phi(t)$ is even we see that $x(-t) = x(t)$ but in general $y(-t) \neq y(t)$. This means that the point $(x(t), y(-t))$ on the curve is obtained as the second intersection point of C with the "vertical" line passed through the initial point (x, y). Thus the transformation $t \to -t$ is equivalent to the involution I_1 in the John algorithm. Quite analogously, the transformation $t \to -t - 2\varepsilon$ remains the coordinate y on the curve C unchanged whereas x is transformed to another point on the intersection of the curve C with "horizontal" line. Thus, transformation $t \to -t - 2\varepsilon$ is equivalent to second involution I_2 in the John algorithm. For the John mapping T we have obviously

$$T = I_2 I_1 \leftrightarrow t \to t - 2\varepsilon, \quad T^{-1} = I_1 I_2 \leftrightarrow t \to t + 2\varepsilon, \tag{3.36}$$

Thus, we proved the following

Proposition 17. For curve $\tilde{C}$ given by (3.34) with an even periodic meromorphic function ϕ satisfying the condition (3.35) the complex John mapping T is equivalent to a shift of the parameter t by the step -2ε.

Now the periodicity condition of the complex John mapping is obtained the form:

$$2n\epsilon = mT \tag{3.37}$$

with some integer n, m.

In the case of the biquadratic curve we have two periods $2\omega_1$, $2\omega_2$ and then the periodicity condition is taken the form:

$$n\epsilon = m_1\omega_1 + m_2\omega_2 \tag{3.38}$$

with some integer n, m_1, m_2.

Now we have only two cases: either all points of C are not periodic (for the John mapping) or each point is periodic with the same period (that is equal to n if the condition (6) or (3.38) is fulfilled).

Consider the last canonical form of previous subsection – the Euler-Baxter curve (3.28) that is parameterized as (3.31). It is important that for given $x = \sqrt{k}\, sn(t;k)$ there are two values of y corresponding to two values of η: $y_1 = \sqrt{k}\, sn(t+\eta;k)$ and $y_2 = \sqrt{k}\, sn(t-\eta;k)$. These two values $y_{1,2}$ correspond to two points of intersection of the line $x = x_0$ with the curve (3.28).

Thus we can find all points M_n of the complex John algorithm:

$$M_n = (\sqrt{k}\, sn(t+2\eta n-3\eta;k),\ \sqrt{k}\, sn(t+2\eta n-2\eta;k)) \tag{3.39}$$

The periodicity condition with a period n is

$$2\eta\, n = 4Km_1 + 2iK'm_2 \tag{3.40}$$

with some integer m_1, m_2 and constants $K = K(k) = \int\limits_0^1 dt/\sqrt{(1-k^2t^2)(1-t^2)}$, $K' = K(k')$, $k'^2 = 1-k^2$ by well-known notations.

In order to write analogous formulae in real case we should write real paramerizations by means of real-valued elliptic functions.

Consider first **the case** $a > 0,\ c = +1$ **of real** x**-**y**−symmetric curve** (3.28) and use the parameterization (3.31). Above, in the proposition 15, (s. 0) we noted that under $a > 0$ the condition $b^2 > (a+1)^2$ (or $\tilde{b} > 2$) is necessary and sufficient for the existence of real curve (3.31). Therefore assume $b^2 > (a+1)^2$ then $k + k^{-1} \geq 2$ and we may choose $k = k_1,\ 0 < k_1 < 1$. From the second equality (3.32) we obtain that $sn(\eta, k)$ is pure imaginary. This means $\eta = 2mK + \theta i$ with an integer m and real θ (see [9]). Moreover, x must be real, that is either $t = nK'i + \tau$ or $t = (2n+1)K + i\tau$ with an integer n and real τ ([9]). But y must also be real, i.e. $t + \eta = K'n_1 i + \tau_1$ or $t + \eta = (2n_1+1)K + i\tau_1$ with an integer n_1 and real τ_1. Let us add expressions for t and η, obtain that it is possible only a variant with the second formulae for t and $t+\eta$: $\tau_1 = \theta + \tau$. Thus we have suppose that the parameter t isn't real: $t = \pm K + i\tau$, however the values x, y in the paramentization (3.31) are real and the parameter τ is real, here the sign before K determines a branch of the curve which is two bounded ovals in this case. The relations (3.32) have a denumerable set of solutions η, we choose $\theta = \mathrm{Im}\ \eta$ as a minimal positive number of all $\mathrm{Im}\ \eta$. Now the periodicity condition (3.40) can be written in the form ($m = m_2$):

$$\frac{\theta}{K'} = \frac{m}{n} \in \mathbb{Q} \tag{3.41}$$

where from relations (3.32) we obtain that the number θ is a minimal positive solution of the equation $\mathrm{sc}(\theta, k') = 1/\sqrt{ak}$ (here and below $sc = sn/cn,\ ns = 1/sn$ and so on as usually). We proved the foolowing

Proposition 18. The condition (3.41) is a criterion of periodicity of the John mapping T in the case $a > 0,\ c = +1$ of real x-y−symmetric curve (3.28). Under it all points of our curve are periodic with the same minimal period n in nonreducible fraction $\frac{m}{n}$. The number m is a number of full turns of the curve that the mapping T^n do.

Consider also **another cases of real** x**-**y**−symmetric curves** (3.28). We take advantage of a list of cases given in the work [35] (note that the last work constains mistakes in above case $c = 1$, $a > 0$, here we used formulae from [10]). We write final parameterization formulae where the modulus k is given by $\hat{b}$ ($k'^2 + k^2 = 1$)) and the shift η − by a or b (they are coordinated). Here $\hat{b} = (b^2 - a^2 - c)/a$.

$$\begin{aligned}&\text{Case } a > 0,\ c = -1, : \quad x = \sqrt{k/k'}\, cn(t;k), \quad y = \sqrt{k/k'}\, cn(t \pm \eta; k),\\ &\text{where } \ k/k' - k'/k = \hat{b}, \quad a = ds^2(\eta,k)/kk',\ \ b = -cs(\eta,k)\, ns(\eta,k)/kk'.\end{aligned} \tag{3.42}$$

$$\begin{aligned}&\text{Case } a<0,\ \ c=-1,: \quad x=\sqrt{k/k'}\,nc(t;k), \quad y=\sqrt{k/k'}\,nc(t\pm\eta;k),\\ &\text{where } \ k'/k-k/k'=\hat b, \quad a=-ds^2(\eta,k)/kk',\ b=cs(\eta,k)\,ns(\eta,k)/kk'.\end{aligned} \tag{3.43}$$

$$\begin{aligned}&\text{Case } a<0,\ c=1,\ \hat b<-2: \ x=\sqrt{1/k'}\,cs(t;k),\ y=\sqrt{1/k'}\,cs(t\pm\eta;k),\\ &\text{where } \ -1/k'-k'=\hat b,\ \ a=-cs^2(\eta,k)/k',\ b=ds(\eta,k)\,ns(\eta,k)/k'.\end{aligned} \tag{3.44}$$

$$\begin{aligned}&\text{Case } a<0,\ c=1,\ \hat b>2:\\ &\qquad\text{unbounded part } x=\sqrt{1/k}\ ns(t;k), \quad y=\sqrt{1/k}\ ns(t\pm\eta;k),\\ &\qquad\text{bounded part } x=\sqrt{k}\ sn(t;k), \quad y=\sqrt{k}\,sn(t\pm\eta;k),\\ &\text{where } 1/k+k=\hat b, \quad a=-ns^2(\eta,k)/k,\ b=cs(\eta,k)\,ds(\eta,k)/k.\end{aligned} \tag{3.45}$$

Separation in the last case is connected with that the curve has a bounded oval and other unbounded branches. Last four cases are given in the work [35].

Remaining cases are considered in the work [36]: Case $a<0$, $c=1$, $|\hat b|<2$ of real symmetric curves (3.28) can be transformed to the case (3.44) by means of substitution

$(x,y)=\left(\dfrac{1-\bar x}{1+\bar x},\dfrac{1-\bar y}{1+\bar y}\right)$. Then, $\bar a=\dfrac{1+a-b}{1+a+b}$, $\bar b=\dfrac{2(1-a)}{1+a+b}$, $\bar{\hat b}=2+\dfrac{16a}{b^2-(a+1)^2}$.

$$\begin{aligned}&\text{Case } a<0,\ c=1,\ |\hat b|<2: \ \bar x=\sqrt{k'}\,sc(t;k),\ \bar y=\sqrt{1/k'}\ sc(t\pm\eta;k),\\ &\text{where } \ -1/k'-k'=\bar{\hat b},\ \ \bar a=-cs^2(\eta,k)/k',\ \bar b=ds(\eta,k)\,ns(\eta,k)/k'.\end{aligned} \tag{3.46}$$

For **real x-y−asymmetric curves (1.4), the case (iii)** (see the statement 7)

$$\begin{aligned}&\text{Case } a>0,\ c=1,\ (\hat b>2): \ x=\sqrt{k'}\,nd(t;k),\ y=\pm\sqrt{1/k'}\ dn(t\pm\eta;k),\\ &\qquad\text{where } \ 1/k+k=\hat b,\ \ a=k'nd^2(\eta,k),\ b=k^2sd(\eta,k)\,cd(\eta,k).\end{aligned} \tag{3.47}$$

$$\begin{aligned}&\text{Case } a<0,\ c=1,\ (\hat b<-2): \ x=\sqrt{k'}\,sc(t;k),\ y=\sqrt{1/k'}\ cs(\pm t+\eta;k),\\ &\qquad\text{where } \ -1/k-k=\hat b,\ \ a=-k'\,nd^2(\eta,k),\ b=k^2sd(\eta,k)\,cd(\eta,k).\end{aligned} \tag{3.48}$$

All these cases (3.42)-(3.48) admit calculations as above that gave us the formula (3.41). Such calculations were done by our collaborators M.V. Belogljadov and A.A. Telitsyna ([11],[56]) and they bring the following results:

a common answer for cases (3.42), (3.44), (3.45), (3.46), (3.48)

$$\frac{\mathrm{Re}\ \eta}{2K}=\frac{m}{n}\in\mathbb{Q}; \tag{3.49}$$

for the case (3.43)

$$\frac{\mathrm{Re}\ \eta}{K}=\frac{m}{n}\in\mathbb{Q}; \tag{3.50}$$

for the case (3.47)

$$\frac{\mathrm{Im}\ \eta}{2K'}-1/2=\frac{m}{n}\in\mathbb{Q}. \tag{3.51}$$

Here k and η are counted up from relations (3.42)-(3.48) corresponding to considered case. Thus, the following proposition takes place:

Proposition 19. The conditions (3.49)-(3.51) are criterions of periodicity of the John mapping T in corresponding real cases (3.42)-(3.48). The number n is the period of dynamical system and m is the number of full turns of mapping T^n.

Note that a correspondence to cases of proposition 15 can be easy seen and presence of several cases corresponding to a case of the proposition 15 is explained by technical reasons (by choice of an infinite point on projective line).

4. The Poncelet problem

In this section we demonstrate a nice correspondence between the mapping by F.John and famous the Poncelet problem (the Poncelet porism) for two conics. We start by recalling the Poncelet porism in a well-known form.

4.1. **The Poncelet porism in form of two circles.** At the beginning let a circle A lie inside another circle B. From any point on B, draw a tangent to A and extend it to B. From the intersection point, draw another tangent, etc. For n tangents, the result is called an n-sided Poncelet transverse. This Poncelet transverse can be closed for one point of origin, i.e. there exists one circuminscribed (simultaneously inscribed in the outer and circumscribed on the inner) n-gon. We could begin with a polygon that is understood as the union of a set of straight lines sequentially jointing a given cyclic sequence of points (vertices) on the plane. If there exist two circles, inscribed and circumscribed for this polygon, then this polygon is called a bicentric polygon. Note that sides of the polygon can intersect and the intersection point is not obligatory to be a vertex. Furthermore, the inscribed circle does not obligatory touch a segment between vertices, the contact point can lie on extension of the side and therefore the circles can intersect. Bicentric polygons are popular objects of investigations in geometry. This is most known form of the Poncelet porism. If we denote by r the radius of the inscribed circle, by R the radius of the circumscribed circle and by d a distance between the circumcenter and incenter for a bicentric polygon then these three numbers can not be arbitrary and together with n they satisfy some relations. So, for the case of triangle the relation is sometimes known as the Euler triangle formula $R^2 - 2Rr - d^2 = 0$. One of popular notations for such relations (necessary and sufficient for existence of a bicentric polygon) is given in terms of additional quantities

$$a = \frac{1}{R+d}\ , \quad b = \frac{1}{R-d}\ , \quad c = \frac{1}{r}\ .$$

So, for a triangle above the Euler formula has the view: $a + b = c$, for a bicentric quadrilateral, the radii and distance are connected by the equation $a^2 + b^2 = c^2$. The relationship for a bicentric pentagon is $4(a^3 + b^3 + c^3) = (a + b + c)^3$. In a general case one introduses numbers

$$\lambda = 1 + \frac{2c^2(a^2 - b^2)}{a^2(b^2 - c^2)}\ , \quad \omega = \cosh^{-1}\lambda\ , \quad k^2 = 1 - e^{-2\omega}, \quad K = K(k) \text{ (see s. 3.5)}$$

and then the relationship can be written by means of elliptic functions in the form

$$\operatorname{sc}\left(\frac{K}{n}, k\right) = \frac{c\sqrt{b^2 - a^2} + b\sqrt{c^2 - a^2}}{a(b+c)} \tag{4.1}$$

(Richelot (1830) – the first edition of the criterion, Kerawala (1947) ([37]) – the written criterion).

4.2. **Setting of the Poncelet problem.** Recall the Poncelet problem [13] for the case of two ellipses, for simplicity and as it was introduced by Jean-Victor Poncelet himself. We take two arbitrary ellipses A and B, A inside B. Let us have an arbitrary point Q_1 on the ellipse A and pass a tangent straight line to A at the point Q_1. This tangent crosses the ellipse B at two points P_1 and P_2, P_1 before P_2 with respect to a standard orientation. Then we take the point P_2 on B and pass the second tangent to the ellipse A. We denote as Q_2 the point on A where this tangent contacts with A. This tangent meets the ellipse B in two points P_2 and P_3. Take the point P_3 and repeat this procedure. Then we obtain a mapping $U_B : B \to B$ which acts by the rule $U_B : P_k \to P_{k+1}$ that will be called the Poncelet mappings below. Moreover, we obtain the mapping $U_A : A \to A$ acting by the rule $U_A : Q_k \to Q_{k+1}$. More precisely, because the definition of U_B exploits a point $Q \in A$ we should introduce two mappings $I_A, I_B : \tilde{C} \to \tilde{C}$, $\tilde{C} := \{(Q, P) \in A \times B \mid P \text{ lies on tangent line to } A \text{ at } Q\}$, acting by the rules $I_A : (Q_1, P_2) \to (Q_2, P_2)$, $I_B : (Q_1, P_1) \to (Q_1, P_2)$. These mappings I_B, I_A generate a composition $U = I_B \circ I_A$ that is similar to the John mapping T. We obtain also two sets of points P_n and Q_n on the ellipses B and A, respectively.

The mapping U_B has an inverse and generates a discrete dynamical system or, in other words, an action of the group $\mathbb{Z}$ on B as above for the John mapping. An orbit of this action is the set of the points P_k, $k = ..., -1, 0, 1, 2, ...$ and $P_k = U_B^{k-1} P_1$. Now note that in a general case of a disposition the ellipses can be intersecting, then we must start from a point Q_1 on the ellipse A which is inside B and can determine the mappings U_B, U_A in the same way. In this case we can encounter with that the tangent straight line intersects the ellipse B only at one point, then we must consider this point as double. The ellipses can be tangent in one or two points, they can be tangent and intersecting simultaneously, they can be nonintersecting and lie one outside other, finally, they can be arbitrary irreducible conics. The way is the same. Note that one can consider the case the conic B is reducible, e.g., it is two nontangential different straight lines. Note more, that each projective transformation of the plane transforms a Poncelet mapping of conics into the same mapping of their immages, therefore we could restrict ourself to a case when one of conics is the unit circle.

The first interesting problem is to describe these crossing points explicitly. It was solved by Jacobi and Chasles who showed that the sequences P_n and Q_n can be parameterized by the elliptic functions. The second problem, the so-called Poncelet porism or the big Poncelet theorem (see [13]), consists in showing that if a particular trajectory of action on conics is closed (i.e., if $P_N = P_0$ for some $N > 2$) then this property does not depend on the choice of the initial point Q_1 on the conic A (or on the choice of P_0). A modern treatment of this problem from an algebro-geometric point of view can be found, e.g., in [28]. Our approach is another. If we introduce the standard rational parameterizations of our conics (see below (4.3), (4.4)) then the parameters x and y of the points Q_1, P_1 prove to be connected by a polynomial equation $F(x, y) = 0$. In a generic situation $F(x, y)$ is a polynomial of the exact degree 2 with respect to each variable. Indeed, for a nondegenerate situation a tangent to the conic A at a point x should intersect B in two distinct points, and, moreover, from a point y on B there are two distinct tangents to A. Conversely, for any polynomial $F(x, y)$ of the degree 2 in x and y it is possible to find two conics A and B parameterized as in (4.3) and (4.4). These arguments are sufficient to build

trajectories of the John dynamical system. Note that similar considerations were exploited in [25] in order to show that the tangent and intersection points belong to an elliptic curve. The authors of [25] also introduced two involutions I_1 and I_2 which are equivalent to our ones in the John T-algorithm for the Euler curve. In our approach we derive the curve $F(x,y)=0$ explicitly and study it. Note connections with Gelfand's question [38] and elastic billiard [58], [41].

4.3. **Passage to the John mapping on a biquadratic curve.** Find an explicit expression for the Poncelet mapping U_B. We introduce the standard rational parameterization of an arbitrary conic [13] that can be found even if the conic is reducible. Assume that the conic A is described by the coordinates ξ_0,ξ_1,ξ_2 of a two-dimensional projective plane, that is

$$\sum_{i,j=0}^{2}\widetilde{A}^{ij}\xi_i\xi_j=0 \tag{4.2}$$

and, as it is well-known, the conic is irreducible iff the matrix $\widetilde{A}$ is nondegenerate. Note that we will describe vectors (contravariant tensors) by lower indexes and covectors (covariant tensors) by upper indexes because we often use indexes for exponents of degree.

Corresponding affine coordinates will be denoted by ξ,η: $[\xi_0:\xi_1:\xi_2]=[1:\xi:\eta]$, of course for $\xi_0\neq 0$. Then it is possible to find polynomials $E_0(x),E_1(x),E_2(x)$ with $\deg(E_i(x))\leq 2$ such that

$$\xi_i=E_i(x),\qquad\text{or}\qquad \xi=\frac{E_1(x)}{E_0(x)},\quad \eta=\frac{E_2(x)}{E_0(x)}. \tag{4.3}$$

Quite analogously, the conic B can be parameterized as

$$\xi_i=G_i(y),\qquad\text{or}\qquad \xi=\frac{G_1(y)}{G_0(y)},\quad \eta=\frac{G_2(y)}{G_0(y)}, \tag{4.4}$$

where $G_i(y)$ are some other polynomials of the degrees not exceeding 2. Thus the value of parameter x completely characterizes the point on the conic A, and the value of y completely characterizes the point on the conic B.

Let us consider more general case now.

Lemma 1. Let curves A and B be given parametrically by (4.3), (4.4) with some smooth functions E_i,G_i and $E_0\not\equiv 0, G_0\not\equiv 0$ on each interval. The point $P=G(y)$ lies on the tangent line to A at $Q=E(x)$ iff $F(x,y)=0$ where

$$F(x,y)=\begin{vmatrix} E_0(x) & E_1(x) & E_2(x)\\ E_0'(x) & E_1'(x) & E_2'(x)\\ G_0(y) & G_1(y) & G_2(y)\end{vmatrix}. \tag{4.5}$$

Proof. The affine tangent line L to the curve A at a point Q with the parameter x has the direction vector $\tau=(\frac{d\xi}{dx},\frac{d\eta}{dx})$. Therefore the affine point P satisfies the equality

$$\overrightarrow{OP}(y)-\overrightarrow{OQ}(x)=k\tau$$

with some $k\in\mathbb{R}$ and an origin O. The last equality is equivalent to the complanarity of vectors $\vec{E}=(E_0(x),E_1(x),E_2(x))$, $\vec{E}'=(E_0'(x),E_1'(x),E_2'(x))$ and $\vec{G}=(G_0(y),G_1(y),G_2(y))$ in the bundle space $\mathbb{R}^3\setminus\{0\}$ of the projective fiber bundle $\mathbb{R}^3\setminus$

$\{0\} \to \mathbb{RP}^2$, where the prime denotes the derivative with respect to x. Indeed, the collinearity of the vectors $\overrightarrow{OP}(y) - \overrightarrow{OQ}(x)$ and τ means

$$0 = \begin{vmatrix} \frac{E_1(x)}{E_0(x)} - \frac{G_1(y)}{G_0(y)} & \frac{E_2(x)}{E_0(x)} - \frac{G_2(y)}{G_0(y)} \\ \left(\frac{E_1}{E_0}\right)'(x) & \left(\frac{E_2}{E_0}\right)'(x) \end{vmatrix} = \begin{vmatrix} 1 & \frac{E_1}{E_0} & \frac{E_2}{E_0} \\ 0 & \frac{E_1}{E_0} - \frac{G_1}{G_0} & \frac{E_2}{E_0} - \frac{G_2}{G_0} \\ 0 & \frac{E_1'}{E_0} - \frac{E_1 E_0'}{E_0^2} & \frac{E_2'}{E_0} - \frac{E_2 E_0'}{E_0^2} \end{vmatrix} =$$

$$= - \begin{vmatrix} 1 & E_1/E_0 & E_2/E_0 \\ 1 & G_1/G_0 & G_2/G_0 \\ E_0'/E_0 & E_1'/E_0 & E_2'/E_0 \end{vmatrix} = \begin{vmatrix} E_0(x) & E_1(x) & E_2(x) \\ E_0'(x) & E_1'(x) & E_2'(x) \\ G_0(y) & G_1(y) & G_2(y) \end{vmatrix} \frac{1}{G_0(y)E_0(x)^2}.$$

□

Return to our case of conics. We see $F(x, y)$ has the form

$$F(x, y) = M_0(x)G_0(y) + M_1(x)G_1(y) + M_2(x)G_2(y) \tag{4.6}$$

with polynomials $M_i(y)$ defined as

$$M_i(x) = \epsilon_{ikl}(E_k'(x)E_l(x) - E_k(x)E_l'(x)), \quad i, k, l = 0, 1, 2, \tag{4.7}$$

where ϵ_{ikl} is the completely antisymmetric tensor. One can easily check that $\deg(M_i(x)) \le 2$, therefore the curve $F(x, y) = 0$ is a biquadratic curve of the form (1.4).

Note that the equality (4.6) can be written as $F(x, y) = (\,\vec{M}(x), \vec{G}(y)\,)$ with the scalar product $(\cdot, \cdot)$ or $F = \langle\, \vec{M}(x), \vec{G}(y)\,\rangle$ with the pairing $\langle \cdot, \cdot \rangle$ (in this case $\vec{G} \in \mathbb{R}^3, \vec{M} \in (\mathbb{R}^3)^*$) and $F = (\vec{E}(x), \vec{E}'(x), \vec{G}(y))$ with the mixed product in $\mathbb{R}^3$. Further, we introduce vectors $\vec{x} = \mathit{colon}(1, x, x^2), \vec{y} = \mathit{colon}(1, y, y^2)$ and matrixes E, G by the rules $E_i(x) = \sum_{j=0}^{2} E_{ij}x^j = (E\vec{x})_i$, $\vec{G} = G\vec{y}$, then the decomposition (4.6) can be written in the form $F(x, y) = (M\vec{x}, G\vec{y})$, where $\vec{M}(x) = M\vec{x}$. This implies that $F = (\vec{x}, M^*G\vec{y}) = (G^*M\vec{x}, \vec{y})$. Comparing with the decomposition in the forms (3.1) and (3.2) we obtain

$$A = G^*M, \ B = M^*G = A^*, \tag{4.8}$$

where the matrix A is obtained in the same way as the matrix E above, moreover $A = (a_{ik})$ with the matrix from (1.4).

In a case of irreducible conics the matrix A is nondegenerate. Indeed, if the matrix A is degenerate then either the matrix M or the matrix G is degenerate by virtue of (4.8). The degeneracy of G means a linear dependence of polynomials $G_i(y)$ that is the conic B will be a straight line in this case. The degeneracy of M means that there exist constants c_0, c_1, c_2 such that

$$\begin{vmatrix} E_0(x) & E_1(x) & E_2(x) \\ E_0'(x) & E_1'(x) & E_2'(x) \\ c_0 & c_1 & c_2 \end{vmatrix} \equiv 0,$$

i.e. a linear dependence of polynomials $E_i(x)$ and the conic A will be a straight line.

Return to the Poncelet construction. We have obtained the parameters x_1 of the point Q_1 and y_1 of the point P_1 satisfy the equation $F(x_1, y_1) = 0$, with F from (4.5). Note now that instead the point P_1 we could write the point P_2 with the parameter y_2 in the Poncelet construction and have the same equation $F(x_1, y_2) = 0$. We obtain the first result: for any point given by a parameter value x_1 on the conic A, the points with parameters y_1 and y_2 of the intersection points

of the tangent line L_1 at x_1 with B are determined as two roots of the quadratic equation: $F(x_1, y) = 0$. Thus identifying a point with its parameter value we can say the Poncelet mapping U_B maps the point y_1 into the point y_2 and hence it (more precisely, the mapping I_B) coincides with the John mapping I_1 from section 2. It is analogously, the mapping U_A maps the point x_1 into the point x_2 and hence it (the mapping I_A) coincides with the John mapping I_2, and the mapping U coincides with the John mapping T on the curve C (1.4).

We have proved the following

Proposition 20. Each pair of distinct irreducible real conics A, (4.3) and B, (4.4) generates a biquadratic real polynomial $F(x, y)$ of the form (1.4) with nondegenerate matrix A by means of (4.5) such that the point $y \in B$ lies on the tangent at the point $x \in A$ if and only if $F(x, y) = 0$. The Poncelet mapping U_B gives us (and can be obtained from) the mapping $U : \tilde{C} \to \tilde{C}$ that coincides with the John mapping T on the curve C (1.4).

Let us prove the following converse

Proposition 21. For any biquadratic real polynomial $F(x, y)$ of the form (1.4) with nondegenerate matrix A and for every its decomposition (4.6) with given polynomials $G_i(y), M_i(x)$ of the second order there exists an unique projective set of polynomials $E_i(x)$ of the second order such that relations (4.7) (and (4.5)) hold, and, hence, we can relate with any such curve $F(x, y) = 0$ and its decomposition (4.6) a pair of conics A and B parameterized as in (4.3) and (4.4).

Proof. Let us have a decomposition (4.6) of a given biquadratic polynomial F. Then we have a parameterization of the conic B by means of projective coordinate G_0, G_1, G_2 and the first our aim is to find polynomials E_0, E_1, E_2 such that the equalities (4.7) hold. In other words, we must find a polynomial solution of the system of ordinary differential equations

$$E \times E' = M \tag{4.9}$$

with a known vector $M = (M_0(x), M_1(x), M_2(x))$, an unknown vector $E = (E_0(x), E_1(x), E_2(x))$ and the vector product $\times$ in $\mathbb{R}^3$. One can interpret this equation as a problem to find a curve (more exactly, the tangent vector of a curve) if it is known its binormal. Now we will need the following

Lemma 2. Consider the differential equation in $\mathbb{R}^3$

$$r \times r' = b \tag{4.10}$$

with a known smooth vector function $b = (b_1(t), b_2(t), b_3(t))$ and unknown vector function $r = (r_1(t),\ r_2(t), r_3(t))$ depending on a real parameter t. If $(b, b', b'') < 0$ then the equation (4.10) has not any smooth solution. If $(b, b', b'') > 0$ then a solution of the equation (4.10) exists there and it is only $r = \pm(b, b', b'')^{-1/2}\, b \times b'$. If $(b, b', b'') = 0$ on an interval then only four cases are possible with some parameter change $s = s(t)$: 1) $r \equiv r_0 = const$, 2) $r = r_0 + r_1 s$, r_0, r_1 are constant vectors. 3) $r = r_0 e^s + r_1 e^{-s}$ and 4) $r = r_0 \cos s + r_1 \sin s$ with the same r_0, r_1.

Proof. Note first that the Jacobi determinant of the system (4.10) of ordinary differential equations with respect to r' is identically equal to zero, so that a standard theory of differential equations systems does not work for this system.

I). Let $b \times b' \not\equiv 0$ on an interval. We observe that the solution must be only of the form $r = v(t)\, b \times b'$ because from (4.10) we easily obtain: $b \cdot r = 0,\ b \cdot r' = 0$, so $b' \cdot r = 0$. The scalar v is unknown still. Substituting such r in (4.10) we obtain $v^2(b, b', b'')b = b$ so that there exists no solution of (4.10) when $(b, b', b'') < 0$. The equality $(b, b', b'') = 0$ on an interval gives $b = 0$, i.e. a contradiction.

II). Let now $b \times b' \equiv 0$ on an interval. If $b \not\equiv 0$ and $b' \not\equiv 0$ then $b(t) = \mu(t) b_0$ with a scalar function μ, b_0 is a constant vector that means the curve r is a plane curve. Note that we examine simultaneously the case $b \not\equiv 0$ and $b' \equiv 0$ ($\mu \equiv 1$). Substitute this b in (4.10) and choose another parameter $\tilde{t}$ such that $r \times r' = b_0$. It follows from this that $r \times r'' = 0$, so $r'' = \nu(\tilde{t})r$ with a scalar function ν. After a reparameterization $\tilde{t} \to s$ we obtain $r'' = \pm r$, i.e. $r = r_0 e^s + r_1 e^{-s}$ or $r = r_0 \cos s + r_1 \sin s$ with some constant vectors r_0, r_1 of a plane which is orthogonal to b_0. For $r_0 \neq r_1$ these solutions are cases of a hyperbolic and an elliptic rotation of the plane pespectively, besides, there are also cases of a movement along a direct straight line (cases $b = 0$ and $r_0 = r_1 \neq 0$) and of a stationary state. $\square$

Continuation of the proposition proof. We will apply the lemma to the case of polynomial vectors of the second order $b = M$ and $r = E$. It is easily seen that for such M the equality $M \times M' \equiv 0$ implies only trivial cases 1) or 2) of the lemma, so that $M \times M' \not\equiv 0$ and $v^2 = (M, M', M'')$. The last mixed product doesn't depend on x. Indeed,

$$\frac{d}{dx}(M, M', M'') = (M', M', M'') + (M, M'', M'') + (M, M', M''') = 0$$

as the vector M consists of quadratic polynomials. Note that we can change a sign of the mixed product (M, M', M'') by a change of the sign of M. Thus, for given M_i from the decomposition (4.6) there exists the scalar v such that $v^2 = (M, M', M'')$, hence the components of the derived vector $E = vM \times M'$ are polynomials of the second order (and the mixed product (E, E', E'') doesn't depend on x). $\square$

4.4. **Projective invariance of the biquadratic curve.** One can consider the vector $E \times E'$ as a covector E^* of a dual space $(\mathbb{R}^3)^*$ with components $E^j(x) = \sum_{i=0}^{2} \widetilde{A}^{ij} E_i(x)$, where the matrix $\widetilde{A}$ is from (4.2), more exactly, E^* is proportional to $E \times E'$. Indeed, by definition we at once obtain $\langle E^*(x), E(x)\rangle \equiv 0$ and $\langle E^*, E'\rangle \equiv 0$. The covector field $E^*(x)$ describes a conic in the dual projective space, the field of tangent lines to A (see, e.g., [27]):

$$\sum_{i,j=0}^{2} \widetilde{A}_{ij} E^i E^j = 0 \tag{4.11}$$

with an inverse matrix $(\widetilde{A}_{ij}) = (\widetilde{A}^{ij})^{-1}$. This point of view gives us a representation

$$F(x, y) = \langle E^*(x), G(y)\rangle \tag{4.12}$$

and help to understand why the equality $E \times E' = M$ with quadratic polynomial vector E implies $M \times M' = E$ in the projective sense as $E^{**} = E$, but it doesn't too help for the case of a curve A of order larger than 2.

Proposition 22. For each pair of different irreducible real conics A, (4.3) and B, (4.4), for any projective transformation L of the projective plane the images LA and LB of conics A and B generate the same equation $F(x, y) = 0$ and the curve (1.4). Two different pairs of conics with the same curve are connected by means of a

projective transformation $A \to P_L A,\ B \to P_L B$ and give different decompositions (4.6).

Proof. In order to derive the first result let remember that an arbitrary nondenerate projective transformation P_L of the affine plane can be obtained from a nondenerate linear transformation L of the bundle space $\mathbb{R}^3 \setminus \{0\}$ of the projective fiber bundle. Such linear transformation L gives a number factor $det\, L$ for F in the formula (4.5). Thus the arbitrary nondenerate projective transformation of the affine plane and conics A and B don't change the equation (1.4).

Further, given decomposition (4.6) can be written in the forms (3.1) and (3.2) and generates the nondegenerate matrixes M, G and then A, B my means of (4.8). Some another decomposition gives matrixes M_1, G_1 and the same F and A: $A = G^* M = A_1 = G_1^* M_1$, so that the linear transformations $L = G_1 G^{-1} = (M_1^{-1})^* M^*$ and $M_1 M^{-1} = (G_1^{-1})^* G^*$ translate the conics A, B into conics $A_1 = LA, B_1 = LB$ respectively as a projective transformations. Note from (4.2) that under transformation L the matrix $\widetilde{A}$ passes into $L^{-1*}\widetilde{A}L^{-1}$ and the matrix M passes into $L^{-1*}M$. ☐

Thus, the biquadratic curve $F(x, y) = 0$ depends only on a projective class of the conics pair. But the conics can be in one of the following generic dispositions (the order is chusen for a correspondence to proposition 15):

0) the conic B lies inside of A, strictly or not, i.e. no tangent line to A cuts B.

1) the conic A lies strictly (i.e. without contact) inside of B so that each tangent line to A cuts B in two different points.

2) the conic A cuts the conic B in four points and there exist a straight line that has no common point with A and B.

3) the conic A intersects the conic B at only two points without contacts.

4) the conic A lies strictly outside of B and the conic B lies outside of A.

5) the conic A cuts the conic B in four points and each straight line has a common point with A or B (there is no common tangent line, a hyperbola and an ellipse).

Now we would like to clarify how one could know by F what case occurs. In order to clarify it let's take the following observations. If the conic A intersects B in a point P then from the point $P \in B$ there is only a tangent line to A, so that in the Poncelet construction there is only a point (x, y) with the parameter y corresponding to $P \in B$ that is the point (x, y) is an y-vertex of C, i.e. an extreme point along direction of real axis y. Remember that by the proposition 14 *for generic cases 1)-5) the roots of $D_2(y_1)$ and only they are y-coordinates of y-vertices. Each such vertex corresponds to a point of intersection $A \cap B$.*

In the same way a common tangent line to conics A and B in the plane (ξ, η) gives us an x-vertex (x_2, y_2) in the plane (x, y) and we have: if $A_2(x_2) = 0$ then $y_2 = -A_0(x_2)/A_1(x_2)$ otherwise $D_1(x_2) = A_1^2(x_2) - 4A_0(x_2)A_2(x_2) = 0$, $y_2 = -A_1(x_2)/2A_2(x_2)$. And *for generic cases 1)-5) real roots of $D_1(x_2)$ and only they are x-coordinates of x-vertices. Each such vertex corresponds to a common tangent line.*

Let us remember that the statement 6 says the invariants g_2 and g_3 of polynomials D_1 and D_2 are the same. Note that the case $\Delta < 0$ (i.e. $k^2 < 0$) corresponds only to the above case 5) because the polynomial D_1 has two real and two complex roots if and only if $\Delta < 0$, therefore if and only if the polynomial D_1 is the same. We obtain the following

Proposition 23. In the case of disposition 0) each of equations $D_1(x_1) = 0$ and $D_2(y_1) = 0$ has no real solution and the curve is not real. In the case 1) each of equations $D_1(x_1) = 0, D_2(y_1) = 0$ has no real solution, the curve C is real and it has no a vertex. In the case 2) each of equations $D_1(x_1) = 0, D_2(y_1) = 0$ has four real solutions, the curve C has four x-vertices and four y-vertices. In the case 3) each of equations $D_1(x_1) = 0, D_2(y_1) = 0$ has two real solutions, the curve C has two x-vertices and two y-vertices. In the case 4) we have four common points, so that the equation $D_1(x_1) = 0$ has four real solutions, the equation $D_2(y_1) = 0$ has no real solution, the curve C has no y-vertex and four x-vertices. In the case 5) we have four common tangent lines, so that the equation $D_1(x_1) = 0$ has no real solution, the equation $D_2(y_1) = 0$ has four real solutions, the curve C has no x-vertex and four y-vertices.

Any point of contact gives a point (x_1, y_1) which is both x-vertex and y-vertex, i.e. $D_1(x_1) = D_2(y_1) = 0$, the conditions (3.18) of singular point are fulfilled, and according to the proposition 7 either the curve is irreducible or x_1 and y_1 are multiple zeros of the discriminants $\tilde{D}_1(x)$ and $\tilde{D}_2(y)$ respectively and this singular point is unique. In both cases we observe either the case III of the John's list of dynamical system behaviors and a breakdown of smoothness of the curve or a case of degeneration. In this work we do not consider these cases. Some examinations are in the work [40].

Remark more that the statement 7 and proposition 15 of the subsection 3.4 imply

Proposition 24. For any two conics that are in one of generic dispositions 1)-5) there are real linear-fractional changes R_1, R_2 of real parameters $x = R_1(\bar{x})$, $y = R_2(\bar{y})$ such that the corresponding reduced biquadratic curve C will be having one of the canonical form (i), (ii) or (iii) of the subsection 3.4. Here the disposition 1) corresponds to the case (iii), $-2 < \hat{b} < 2$; the disposition 2) corresponds to the case (iii), $\hat{b} > 2$; the disposition 3) corresponds to the case (iii), $\hat{b} < -2$; the disposition 4) corresponds to the case (i), $c = 1$, $\tilde{b} > 2$; the disposition 5) corresponds to the form (ii), $c = -1$; the disposition 0) corresponds to the case (i), $a > 0$, $c = 1$ with the condition of disappearence $\tilde{b} < 2$ i.e. $b^2 < (a+1)^2$.

4.5. **Periodicity of Poncelet mappings.** Now we can apply the John mapping T in order to construct two sequences of points x_n and y_n on conics A and B. According to proposition 20 of the subsection 4.3 the Poncelet mapping U_B gives us (and can be obtained from) the mapping $U : \tilde{C} \to \tilde{C}$ that coincides with the John mapping T on the curve C (1.4). Therefore the periodic trajectories in the Poncelet problem correspond to closure orbits of the John dynamical system. Thus we obtain main result

Proposition 25. The Poncelet problem in generic setting is periodic if and only if the John mapping on corresponding biquadratic curve (4.5) is periodic and then their periods are coincide.

Explicit answers for the John mapping are given in propositions 15, 18, 19. We obtain

Proposition 26. Dispositions 1)-5) of previous subsection correspond to cases 1)-5) of the proposition 15. The conditions (3.41), (3.49)-(3.51) are criterions of periodicity of the Poncelet mapping U in corresponding cases.

Note that last proposition implies the statement of the big Poncelet theorem.

Since the Poncelet problem is invariant under the arbitrary projective transformation of the plane (ξ, η) we can reduce the conics A and B to some simple shapes. Consider several possibilities.

(i) If we reduce A and B to concentric quadrics determined by the equations

$$\xi^2/a_1 + \eta^2/b_1 = 1, \quad \xi^2/a_2 + \eta^2/b_2 = 1.$$

In this case we have the parameterization

$$\xi = 2a_1^{1/2} y/(1+y^2), \quad \eta = b_1^{1/2}(1-y^2)/(1+y^2)$$

for the conic A and

$$\xi = 2a_2^{1/2} x/(1+x^2), \quad \eta = b_2^{1/2}(1-x^2)/(1+x^2)$$

for the conic B. It is easily verified that the polynomial $Z(x, y)$ defined by (4.5) is reduced in this case to the simplest Euler-Baxter form:

$$Z = x^2y^2 + 1 + a(x^2+y^2) + 2bxy$$

with complex coefficients.

Hence, we have a simple form solution of the Poncelet problem in this case as above (see (3.39)):

$$x_n = \sqrt{k}\,\mathrm{sn}(h(n+s_0); k), \quad y_n = \sqrt{k}\,\mathrm{sn}(h(n+s_0+1/2); k). \tag{4.13}$$

Parameters a_i and b_i can still be specialized further. For instance, it is possible to choose $a_2 = b_2 = 1$ reducing B to the unit circle. This choice corresponds to so-called Bertrand model of the Poncelet process [50]. If the second conic is an ellipse $\xi^2/a_1^2 + \eta^2/b_1^2 = 1,\ a_1 > 1,\ b_1 < 1$ then the formula (4.5) gives us a bounded curve of the form

$$x^2y^2 + \frac{1+b_1}{1-b_1}(x^2+y^2) - \frac{4a_1}{1-b_1}xy + 1 = 0. \tag{4.14}$$

In this case the points x_n are isomorphic to the godograph distribution of spins in the classical XY-chain (see section 7.1 and [26]). Another possible choice $a_2 - a_1 = b_2 - b_1$ corresponds to the confocal quadrics. In this case the Poncelet problem is equivalent to the elastic billiard [58], [41].

(ii) Let us consider a possibility to reduce the conics B and A to two parabolas in the euclidean plane (ξ, η). One of them, say B, can be fixed by the choice $\xi = x, \eta = x^2$, whereas another parabola remains arbitrary: $\xi = F_1(y)/F_0(y),\ \eta = F_2(y)/F_0(y)$, where $F_0(y) = (ay+b)^2$ is a square of the linear function (this is a characteristic property of any parabola). Then we get

$$Z(x, y) = x^2F_0(y) - 2xF_1(y) + F_2(y). \tag{4.15}$$

Performing additional projective (complex) transformation of the variable y we can fix polynomials $F_1(y), F_2(y)$, and $F_0(y)$ in such a way that the polynomial $Z(x, y)$ becomes symmetric in x, y. Then we can reduce $Z(x, y)$ to the form

$$Z = (xy + (x+y)y_0 + g_2/4)^2 - (x+y+y_0)(4xyy_0 - g_3), \tag{4.16}$$

where g_2, g_3 are two remaining (arbitrary) independent parameters of the polynomial Z.

In this case we obtain the Poncelet points parameterized by the Weierstrass function

$$x_n = A_1\wp(h(n+s_0)) + A_0, \quad y_n = A_1\wp(h(n+s_0+1/2)) + A_0.$$

As we will see in Sect. 7, the biquadratic curve of similar form appears for the phase portrait of the elliptic solution of the Toda chain.

Finally, we note that when $F_0 = const$ and $F_1(y)$ is a linear function in y then the parabola A has its axis parallel to that of the parabola B. But then the curve $Z(x, y) = 0$ describes arbitrary conics in the coordinates x and y with absent x^2y^2, x^2y, and xy^2 terms.

4.6. **Cayley determinant criterion.** Let A, B be arbitrary conics as described in previous section. Recall, that all tangents are passing through points of the conic A, whereas all vertices lie on conic B. (It is possible to assume that A is located inside the conic B). Let M_A and M_B are 3×3 matrices describing conics (i.e. corresponding quadratic forms) in projective coordinates x_0, x_1, x_2. I.e. if the conic A is the unit circle $x^2+y^2 = 1$ and conic B is concentirc circle with the radius R then quadratic forms for A, B are

$$x_1^2 + x_2^2 - x_0^2 \quad \text{and} \quad x_1^2 + x_2^2 - R^2x_0^2. \tag{4.17}$$

Corresponding matrices M_A, M_B are diagonal:

$$M_A = diag(1, 1, -1), M_D = diag(1, 1, -R^2).$$

Compute the characteristic determinant

$$F(z) = \det(A - zB). \tag{4.18}$$

Clearly, $F(z)$ is a cubic polynomial. This polynomial is invariant with respect to any similarity transformation $A \to S^{-1}M_AS,\ B \to S^{-1}M_BS$ with a nondegenerate matrix S. As well known, for a pair of quadratic forms, in general, apart from degenerate cases, there exist a transformation reducing both forms simultaneously to a diagonal form. The roots $z_i, i = 1, 2, 3$ of the polynomial $F(z)$ have simple meaning. If the matrix M_B is reduced to identity matrix (i.e. $M_B = diag(1, 1, 1)$), then z_1, z_2, z_3 are diagonal elements (eigenvalues) of the matrix A.

The first step in the Cayley criterion is Taylor expansion of the square root of the polynomial $F(z)$:

$$\sqrt{F(z)} = c_0 + c_1z + c_2z + \dots + c_nz^n + \dots \tag{4.19}$$

Then we compute the Hankel-type determinants:

$$H_p^{(1)} = \begin{vmatrix} c_3 & c_4 & \dots & c_{p+1} \\ c_4 & c_5 & \dots & c_{p+2} \\ \dots & \dots & \dots & \dots \\ c_{p+1} & c_{p+2} & \dots & c_{2p-1} \end{vmatrix}, \quad p = 2, 3, 4, \dots \tag{4.20}$$

and

$$H_p^{(2)} = \begin{vmatrix} c_2 & c_3 & \dots & c_{p+1} \\ c_3 & c_4 & \dots & c_{p+2} \\ \dots & \dots & \dots & \dots \\ c_{p+1} & c_{p+2} & \dots & c_{2p} \end{vmatrix}, \quad p = 1, 2, 3, \dots \tag{4.21}$$

Then the Cayley criterion [13], [28] is:

Statement 8. The trajectory of the Poncelet problem is periodic with the period N if and only if $H_p^{(1)} = 0$, if $N = 2p$ and $H_p^{(2)} = 0$, if $N = 2p + 1$. For modern proof of the Cayley criterion see [28].

Illustration. Take again the simplest case of two circles (4.17) with radii 1 and R. Then obviously,

$$F(z) = (z-1)^2(zR^2-1). \tag{4.22}$$

The first nontrivial Taylor coefficients of $\sqrt{F(z)}$ are $c_2 = R^2(R^2-4)/8$ and $c_3 = R^4(R^2-2)/16$. The case $c_2 = 0$ means $R = 2$ which corresponds to perfect triangle trajectory. The case $c_3 = 0$ means $R = 2^{1/2}$ which corresponds to square trajectory.

5. The Pell-Abel equation

5.1. **Historical notes on the Pell-Abel equation and some equivalent problems of analysis.** The Pell equation

$$P^2 - RQ^2 = L \tag{5.1}$$

is a well-known Diophantine equation where for given integer number R which isn't a square, one must find integer P, Q, L satisfying the equation. It is established that L.Euler linked it to Pell by misunderstanding. Actually the equation (5.1) arose and was being studied in works of Indian mathematician Brahmagupta as far back as 1000 years before Euler (628 year) and also in works of Euler's predecessors (among them is P.Fermat). Theory of the Pell equation is well-known ([45]), note only that the standard theory of the equation (5.1) establishes a connection with continued fraction of the number $\sqrt{R}$.

The equation (5.1) in a ring of polynomials $\mathbb{R}[t]$ or $\mathbb{C}[t]$ of one variable with a constant L is called the Pell-Abel equation (it may occur the name "Abel equation" and also "Pell equation for polynomials"). This equations arose in Abel's work of 1826 year where he studied representation of the primitive $\int \rho(t)/\sqrt{R(t)}\,dt$ in elementary functions, here ρ, R are polynomials. N.H. Abel proved that if this primitive can be represented by logarithm and rational functions of t and $\sqrt{R}$ then one can find polynomials P, Q and a number A such that

$$\int \frac{\rho}{\sqrt{R}} dt = A \ln \frac{P + \sqrt{R}Q}{P - \sqrt{R}Q}. \tag{5.2}$$

Here the degree of R is even: $\deg R = 2m$, $\deg \rho = m-1$ and $\rho/A = 2P'/Q$. A main point of Abel's considerations is that the polynomials P and Q satisfy the equation (5.1) with $L = 1$. And conversely, if the polynomials P and Q satisfy the equation (5.1) with $L = 1$ then the equality (5.2) holds with $\rho = 2P'/Q$ and $A = 1$.

Thus, the solvability of the Pell-Abel equation plays a role a integrability criterion of Abel differential. It is well-known that later J. Liouville, V.V. Golubev and successors shown if the integral in the left-side part of (5.2) with some ρ and R can be written by elementary functions then the right-side part must have the view (5.2) with some P and Q and the same ρ. Note also, Abel gave one more criterion for the representation (5.2), namely the formula (5.2) holds iff polynomial continued fraction of the function $\sqrt{R}$ is periodic. Thus, the solvability of the Pell-Abel equation plays also a role of a criterion of periodicity of continued fraction.

Consider one more classical problem – Chebyshev's problem on searching of a polynomial of least deviation. Let us have a system of l closed intervals of real axe $I = [-1,1] \setminus \bigcup_{j=1}^{l-1} (a_j, b_j)$ and the following problem. One should find a polynomial of given degree n with unit leading coefficient that has a least deviation on the set I, i.e. find a minimum of the functional $\|t^n - P_{n-1}(t)\|_{C(I)}$. A general polynomial

$P_{n-1}(t)$ runs a finite-dimensional lineal space and the problem can be interpreted as a problem to find an element in a finite-dimensional lineal subspace of a Banach space which is nearest to a given. But because the space $C(I)$ isn't reflexive such nearest element isn't obliged to exist.

By a method going back to P.L. Chebyshev, one succeeded in proving the existence of such minimal polynomial [2]. Further, if the polynomial P_{n-1} is minimal on the set I then, possible, it will also be minimal on some greater closed subset $E \subset [-1,1]$. Such most wide subset $E = [-1,1] \setminus \bigcup_{j=1}^{m-1} (\alpha_j, \beta_j)$ is called n-right extension of the set I [51]. If now one take a polynomial R in the form $R = (t^2-1) \prod_{j=1}^{m-1} (t-\alpha_j)(t-\beta_j)$ then it is proved to be that the solvability of the Pell-Abel equation (5.1) with unknowns $P(t), Q(t)$, $L = \text{const} > 0$ is equivalent to that the set E is n-right extension of the set I. In addition the polynomial P gives a solution of extremal problem, and the number $\sqrt{L}$ is a least deviation (minimum of the norm) [51].

It is interesting that in this case the subset E is continuous spectrum (which is absolutely continuous and two-valued) of some bi-infinite Jacobi (3-diagonal) selfajoint real periodic matrix in the space l^2 if and only if the set E is n-right or that is equivalently if the Pell-Abel equation (5.1) is solvable (see references in [51]). Remark that the Chebyshev's problem will be got for the case of one interval $E = [-1,1]$, $R = (1-t^2)$, and the Akhiezer's problem – for the case of two intervals ($\deg R = 4$) and polynomial of the fourth order $E = [-1,a] \cup [b,1], a < b$, $R = (1-t^2)(t-a)(t-b)$. Note that there are several solvability criterions for the Pell-Abel equation with a polynomial corresponding to the Akhiezer's problem, among them is well-known Zolotarev's porcupine (see, for instance, [43]). Note also recent Khrushchev's work [39] on links to continued fractions.

5.2. **Connection between the Poncelet problem and the Pell-Abel equation.** The solvability of the Pell-Abel equation

$$P(t)^2 - R(t)Q(t)^2 = L^2 \tag{5.3}$$

with the polynomial R of even order, as we have noted above, has several equivalent formulations ([51]). In the work of V.A. Malyshev [43] there is a new solvability criterion given in an algebraic form that we will formulate for interesting for us case of the fourth order $R = t^4 + d_1 t^3 + ... + d_4$. Let us expand the root $\sqrt{R}$ in Laurent series in a neighborhood of infinity:

$$\sqrt{R} = \sum_{j=-m}^{\infty} C_j t^{-j} \tag{5.4}$$

and make up a determinant of Hankel type:

$$\Gamma_k = \begin{vmatrix} C_1 & C_2 & \dots & C_k \\ C_2 & C_3 & \dots & C_{k+1} \\ \dots & \dots & \dots & \dots \\ C_k & C_{k+1} & \dots & C_{2k-1} \end{vmatrix}, \quad k = 1,2,3,\dots \tag{5.5}$$

Malyshev criterion ([43]) states:

Statement 9. The Pell-Abel equation (5.1) with polynomial R of the fourth order has some polynomials P and Q of orders $k+2$ and k as a solution if and only if $\Gamma_k = 0$.

Our observation is following. Let consider the Pell-Abel equation (5.3) with R of the fourth order and let λ_1 be one of roots of polynomial $R(t)$. Make the shift of parameter $t \to t+\lambda_1$ (i.e. $t = \tilde{t} + \lambda_1$). New Pell-Abel equation (5.3) will also be solvable. Now apply the Malyshev's criterion to new equation and let the coefficients C_j be coefficients of Laurent series of the root $\sqrt{R(t+\lambda_1)}$. Then the equality $\Gamma_k = 0$ is necessary and sufficient for solvability of the equation (5.3) with given orders of polynomials P, Q. On the other hand, let in initial setting make change of variable $s = 1/(t-\lambda_1),\ t = \lambda_1 + 1/s$. Then we obtain

$$R(t) = F(s)/s^4 \tag{5.6}$$

with a polynomial $F(s)$ of the third order and therefore $\sqrt{R} = \sqrt{F(s)}/s^2$. Note that the polynomial R can be restored from F by inverse transformation:

$$R = t^4 F(1/t) = t(z_1 t - 1)(z_2 t - 1)(z_3 t - 1), \tag{5.7}$$

where z_1, z_2, z_3- roots of the polynomial F of the third order. The polynomial F can be generated by formula $F = \det(A - zB)$ and matrixes $A = diag(z_1, z_2, z_3)$, $B = diag(1, 1, -1)$ that are built by quadratic forms $z_1x_1^2 + z_2x_2^2 + z_3x_0^2$ and $x_1^2 + x_2^2 - x_0^2$ of conics

$$z_1x^2 + z_2y^2 + z_3 = 0, \quad x^2 + y^2 = 1. \tag{5.8}$$

The expansion (4.19) of the root $\sqrt{F(s)}$ gives us the expansion $\sqrt{F(s)}/s^2 = c_0/s^2 + c_1/s + c_2 + \dots + c_n s^{n-2} + \dots$ which after inverse change can be written as

$$\sqrt{R(t+\lambda_1)} = c_0t^2 + c_1t + c_2 + c_3/t + \dots + c_n/t^{n-2} + \dots.$$

We see that $C_1 = c_3, C_2 = c_4, ...$ and that the determinant (4.20) coincides with the determinant (5.5), i.e. the Cayley criterion for the case of the even period with the polynomial F coincides with the Malyshev's criterion for the equation (5.3) with the polynomial R. We have proved

Proposition 27. Let the Poncelet problem for given conics A and B generate a polynomial F of the third order by formula (4.18) and a polynomial R by formula (5.6). Then the equation (5.3) with polynomial R of the fourth order is solvable if the Poncelet problem is periodic with an even period. Vice versa, for given real polynomial R of the fourth order without constant term one can build a pair of conics A and B so that the Poncelet problem (generated by polynomial F by means of formulae (4.18) and (5.6)) is periodic with an even period if the equation (5.3) with polynomial R is solvable.

Now the proposition 19 gives us solvability criterions for the Pell-Abel equation (5.3).

6. Criterions of uniqueness breakdown for the Dirichlet problem and Ritt's problem

In this section we give criterions of solution uniqueness for the Dirichlet problem (1.1),(1.2) and show how to construct a system of some special solutions in case of non-unique the Dirichlet problem for the string equation. We describe a method that can be applied to a slightly more general class of curves than biquadratics.

We'll consider the Dirichlet problem (1.1),(1.2) in the classical setting (2.2) if the curve C is bounded and in modified setting (2.3) if the curve C is unbounded because we'll need an application of the proposition 1.

Remind that in subsection 3.5 we consider an even periodic meromorphic function $\phi(z)$ with some (complex) period T and a complex curve $\tilde{C} \subset \mathbb{C}^2$ described parametrically as

$$x(t) = \phi(t), \quad y(t) = \phi(t+\varepsilon), \quad t \in \mathbb{C} \tag{6.1}$$

with the property (3.35), ε is a nonzero complex parameter. We considered also a real curve $C = \tilde{C} \cap \mathbb{R}^2$ given by means of a contraction of the functions x, y on a set $S \in \mathbb{C}$. We saw that the curve $\tilde{C}$ is symmetric with respect to the line $y = x$, the complex John mapping T is equivalent to a shift of the parameter t by the step -2ε and the periodicity condition of the complex John mapping has the form: $2n\epsilon = mT$ with some integer n, m.

We assume also (this is a very strong restriction, as we will soon see) that the function $\phi(z)$ possesses a nontrivial multiplication property:

$$\phi(nz) = R_n(\phi(z)), \ n = 1, 2, 3, \dots \tag{6.2}$$

where $R_n(z)$ is a family of rational functions of the argument z (by definition, $R_1(z) = z$, but for other $n = 2, 3, \dots$ the expression and even degree of $R_n(z)$ can be non-obvious). We will consider the condition

$$\forall x \in \mathbb{R}, \quad R_n(x) \neq \infty \tag{6.3}$$

Consider the complex setting (2.7) of homogeneous Dirichlet problem. Now the sufficient condition of the uniqueness of the Dirichlet problem from proposition 2 requires the transitivity of T that implies that for each integers n and m we have $2n\varepsilon \neq mT$.

Assume now that this condition isn't fulfilled, i.e. for some integers N, M we have the condition

$$2N\varepsilon = MT \tag{6.4}$$

We show that in this case the problem is indeed non-unique and we construct explicitly a system of explicit solutions

$$\Phi_n(x, y) = f_n(x) + g_n(y), \ n = 1, 2, \dots \tag{6.5}$$

for the string equation in the domain bounded by the curve C.

As the rational functions $R_n(z)$ are assumed to be non zero, bounded on $\mathbb{R}^2$ and non-constant then the function $\Phi(x, y) = f_n(x) + g_n(y) = R_{2nN}(x) - R_{2nN}(y)$ is obviously nonzero and smooth in $\mathbb{R}^2$. So we should verify the Dirichlet boundary condition $\Phi(x, y) \equiv 0$ in all points of the curve C. Indeed, for any point t in the curve we have

$$f_n(x(t)) + g_n(y(t)) = R_{2nN}(x(t)) - R_{2nN}(y(t)) = \phi(2nNt) - \phi(2nNt - 2nN\varepsilon) = 0,$$

where we exploited properties (3.34), (6.2) and (6.4). Hence the function $\Phi_n(x, y)$ is identically zero in all points on the curve C.

We proved the following

Proposition 28. Under the condition (6.4) for the curve C we can choose

$$f_n(x) = R_{2nN}(x), \quad g_n(y) = -R_{2nN}(y), \tag{6.6}$$

where $R_n(z)$ are rational functions defined by (6.2) with property (6.3). This will provide a non-zero solution (6.5) of the Dirichlet problem in the complex setting

(2.7). If, in addition, the condition (6.3) is fulfilled then obtained solution satisfies the modificed setting (2.3) and if furthermore the curve is bounded then we have a classical solution in interior of the curve C, i.e. in the setting (2.2).

We first illustrate how this theorem works in simplest case when the curve C is ellipse. In this case we can choose parameterization

$$x(t) = \cos(t), \quad y(t) = \cos(t + \varepsilon) \tag{6.7}$$

with some parameter ε. Note that the ellipse described by (6.7) has semiaxis $a = \sqrt{2}\cos(\varepsilon/2), b = \sqrt{2}\sin(\varepsilon/2)$ one of which is inclined by the angle $\pi/4$ with respect to axis x.

Now the John criterion (6.4) of uniqueness of solution is

$$\varepsilon \neq N\pi/M. \tag{6.8}$$

Otherwise, assume that $\varepsilon = N\pi/M$ for some positive integers N, M. The function $\cos z$ satisfies all needed conditions, moreover, we have

$$\cos(zn) = T_n(\cos(z)),\ n = 1, 2, \ldots \tag{6.9}$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind, hence the proposition 28 gives us explicit solutions in this case

$$f_n(x) = T_{2Nn}(x),\ g_n(y) = -T_{2Nn}(y),\ n = 1, 2, \ldots \tag{6.10}$$

Thus, in case of nonuniqueness of the Dirichlet problem for ellipse we get a denumerable family of polynomial nonzero solutions inside the ellipse.

Let C be the biquadratic curve. In beginning, let's consider the case (3.42) of subsection 3.5: $a > 0,\ c = -1$. Here the curve is parameterized as

$$x = \phi(t) = \sqrt{\frac{k}{k'}}\, cn(t;k), \quad y = \phi(t+\eta) = \sqrt{\frac{k}{k'}}\, cn(t+\eta;k).$$

An easy analysis shows the last curve will be real for real t and real η. For the elliptic cosine there are well-known multiplication formulae

$$cn(2mz) = T^1_m(cn^2 z), \quad cn((2m+1)z) = cn\, z\, T^2_m(cn^2 z)$$

with some rational functions $T^1_n(z), T^2_n(z)$. Poles of the function $cn(z)$ lie on the line $\operatorname{Im} t = K'$ therefore the condition (6.3) holds, the curve C is bounded in this case and the proposition 28 states the condition (3.49) is sufficient for uniqueness breakdown for the Dirichlet problem in classical setting. Necessity of the condition (3.49) follows from the proposition 1 (for details see [56]).

In similar ways all another cases can be considered (it done in works [18],[11],[56]). We formulate final result:

Proposition 29. The conditions (3.49)-(3.51) are criterions of uniqueness breakdown for the Dirichlet problem (1.1),(1.2) in corresponding cases (3.42)-(3.48) of the biquadratic curve in a canonical form. The solutions are understood in modificed setting (2.3) of the Dirichlet problem and for the cases of bounded curve $C-$ in classical setting (2.2). Under the condition there is a denumerable set of linear independed analytical solutions of the homogeneous Dirichlet problem.

Now we can consider the problem: what is the most general class of the even periodic functions $\phi(z)$ with the property (6.2)? This problem has solved by Ritt in 1922 [49]. The list of possible such functions includes two generic cases:

(i) $\phi(z) = \frac{a\cos(z)+b}{c\cos(z)+d}$ with arbitrary a, b, c, d;

(ii) $\phi(z)$ is generic even elliptic function of second degree, i.e. $\phi(z) = \frac{a\wp(z)+b}{c\wp(z)+d}$, or, equivalently $\phi(z) = \kappa\,\frac{\sigma(t-e_1)\sigma(t+e_1)}{\sigma(t-e_2)\sigma(t+e_2)}$ with arbitrary parameters $a, b, c, d, \kappa, e_1, e_2$. Parameters g_2, g_3 of the Weierstrass functions are arbitrary (equivalently, periods $2\omega_1, 2\omega_2$ are arbitrary).

There are also 3 exceptional cases connected with elliptic functions with certain restrictions upon their parameters:

(iii) $\phi(z) = \frac{a\wp^2(z)+b}{c\wp^2(z)+d}$ with arbitrary a, b, c, d but with restriction $g_3 = 0$ and g_2 arbitrary real. In this case the fundamental parallelogram of periods $2\omega_2, 2\omega_3$ is a simple square. Such case corresponds to so-called *lemniscatic* elliptic functions.

(iv) $\phi(z) = \frac{a\wp^3(z)+b}{c\wp^3(z)+d}$ with arbitrary a, b, c, d but with restriction $g_2 = 0$ and g_3 arbitrary real. In this case the period lattice has the hexagonal symmetry. Such case corresponds to so-called *equianharmonic* elliptic functions.

For the equianharmonic case Ritt found one more case when $\phi(z)$ is a linear rational combination of the Weierstrass function $\wp'(z)$, but in this case the function $\phi(z)$ is not even and we can omit this case.

The case (ii) corresponds to our problem in biquadratic. Thus, in the case of nonuniqueness we have the solution

$$f_n(x) = R_{2nN}(x), \quad g_n(y) = -R_{2nN}(y), \tag{6.11}$$

where $R_n(z)$ are rational functions defined as (6.2). Note that in contrast to the trigonometric case (i.e. $\phi(z) = \cos(z)$) there are no simple explicit expressions for $R_n(z)$ for arbitrary n.

What about cases (iii) and (iv)? In these cases we have that the curve C is described by an algebraic equation of degree greater than 4. Consider, e.g. the case (iii) and assume for simplicity that $\phi(z) = \wp^2(z)$. We are seeking an algebraic equation $f(x, y) = 0$ for the curve C described parametrically as $x(t) = \wp^2(t),\ y(t) = \wp^2(t + \epsilon)$. This equation can be easily found from addition theorem for the Weierstrass function. We will not write down it explicitly, let us mention only that degree of this equation is 8. Analogously, for the case (iv) we obtain the algebraic curve describing by a polynomial $F(x, y)$ of a total degree 12. But we should note that the John condition of intersection at no more than two points with vertical and horizontal lines is not satisfying and here we do not have a sufficient condition of solution uniqueness for boundary value problems.

7. Related problems of mathematical physics

7.1. **Classical Heisenberg XY spin chain.** There is an interesting relation with the classical Heisenberg XY spin chain [26] which is a system of 2-dimensional unit vectors ("spins") $\vec{r}_n = (x_n; y_n)$, $|\vec{r}_n| = 1$ with energy of interaction

$$E = \sum_{n=0}^{N-1} (\vec{r}_n, J\vec{r}_{n+1}), \tag{7.1}$$

where $J = diag(J_1, J_2)$ is a diagonal 2×2 matrix. The problem is to find static solutions providing local extremum to the energy E. As was shown in [26] this problem is equivalent to finding solutions of the systems of non-linear vector equations:

$$(\vec{r}_n, J(\vec{r}_{n-1} + \vec{r}_{n+1})) = 0, \quad n = 1, 2, \dots, N-1. \tag{7.2}$$

Below we would like to select cases of closed chain when $\vec{r}_0 = \vec{r}_N$ for some N. In what follows we will assume that $J_1 = 1, J_2 = j > 1$. Among all solutions of (7.2) we choose so-called regular ones [26] with the condition

$$\vec{r}_{n-1} + \vec{r}_{n+1} \neq 0. \tag{7.3}$$

Then it is possible to show that the scalar product

$$W = (\vec{r}_n, J\vec{r}_{n+1}) \tag{7.4}$$

doesn't depend on n and hence can be considered as an integral of the system (7.2). It is enough to construct general regular solutions [26] which are of two types. Choice of solution depend on the value of integral W. If $|W| < 1/j$ then

$$x_n = cn(q(n-\theta); k), \quad y_n = sn(q(n-\theta); k), \tag{7.5}$$

where parameters k, q can be found from

$$dn(q; k) = 1/j, \quad k^2 = \frac{1 - j^{-2}}{1 - W^2}. \tag{7.6}$$

If $1/j < |W| < 1$ then

$$x_n = dn(q(n-\theta); k), \quad y_n = k\, sn(q(n-\theta); k), \tag{7.7}$$

where

$$cn(q; k) = 1/j, \quad k^2 = \frac{1 - W^2}{1 - j^{-2}}. \tag{7.8}$$

In both cases parameter θ is arbitrary real number depending on initial condition. If the chain is periodic then we have

$$qN = 4Km_1 + 2iK'm_2 \tag{7.9}$$

which coincide with (3.40). The reason for such coincidence is the following.

Consider the equation of the integral

$$x_n x_{n+1} + j^{-1} y_n y_{n+1} = W \tag{7.10}$$

with fixed value W. The equation (7.10) can be reduced to the algebraic form (3.28) by standard substitution (stereographic projection from the unit circle to the line):

$$x_n = \frac{1 - u_n^2}{1 + u_n^2}, \quad y_n = \frac{2u_n}{1 + u_n^2}$$

It is easily seen that the variables u_n, u_{n+1} lie on the Euler-Baxter biquadratic curve

$$u_n^2 u_{n+1}^2 + 1 + a(u_n^2 + u_{n+1}^2) + b u_n u_{n+1} = 0$$

with parameters a, b simply related to the "physical" parameters j, W.

Then it is easily verified that finding solutions (step-by-step) of equations (7.2) for the regular solutions are equivalent to finding points $M_2, M_3, \dots M_N$ for the John algorithm. Note that arguments similar to ones in the formula (3.41) give us that the parameters q and θ must be real so that the condition (7.9) takes the form

$$\frac{q}{4K} = \frac{m_1}{N}. \tag{7.11}$$

Thus static regular solutions for the closed finite classical Heisenberg XY-chain are equivalent to periodic solutions of the John algorithm for the Euler-Baxter biquadratic curve, or, equivalently, to finding periodicity condition of the Poncelet process for the unit circle $x^2 + y^2 = 1$ and a concentric conic $x^2/\xi_1 + y^2/\xi_2 = 1$. This choice of conics corresponds to the Bertrand model of the Poncelet process

[50]. This means that there is an equivalence between static periodic solutions of the Heisenberg XY-chain and the Bertrand model of the Poncelet process.

7.2. **Elliptic solutions of the Toda chain and biquadratic curve.** Now we consider the Toda chain that is a discrete dynamical system consisting of two sets $u_n(t), b_n(t)$ of complex variables depending on continuous parameter t and discrete parameter $n = 0, \pm 1, \pm 2, \ldots$. The equations of the motion are [57]

$$\dot{b}_n = u_{n+1} - u_n, \quad \dot{u}_n = u_n(b_n - b_{n-1}) \tag{7.12}$$

The Toda chain is one of the most simple and famous completely integrable discrete dynamical systems (for history of this model and review of different approaches see [57]). Among numerous explicit solutions there are so-called "elliptic waves" constructed firstly by Toda himself [57]. We give the Toda elliptic solutions in somewhat different form which is more convenient for applications (Toda used Jacobi elliptic functions whereas we exploit the Weierstrass functions).

Proposition 30. Elliptic solution for the unrestricted Toda chain can be presented in the from

$$\begin{aligned}
&b_n = \omega\Big(\zeta(\omega t - p(n+1) + r) - \zeta(\omega t - pn + r)\Big) - \lambda,\\
&u_n = \omega^2\Big(\wp(p) - \wp(\omega t - pn + r)\Big) =\\
&\omega^2\,\frac{\sigma(\omega t - p(n+1) + r)\,\sigma(\omega t - p(n-1) + r)}{\sigma^2(p)\,\sigma^2(\omega t - pn + r)}
\end{aligned} \tag{7.13}$$

with arbitrary parameters ω, p, r, λ and arbitrary invariants g_2, g_3. Here $\wp(z), \sigma(z)$, $\zeta(z)$ are standard Weierstrass functions defined as in [60].

Proof. The Toda chain equations (7.12) are verified directly by substitution using well known formulas [60]:

$$\frac{d}{dz}\zeta(z) = -\wp(z), \quad \frac{d}{dz}\lg\,\sigma(z) = \zeta(z), \quad \wp(x) - \wp(y) = \frac{\sigma(y-x)\sigma(x+y)}{\sigma^2(x)\sigma^2(y)}$$

Strictly speaking, the variable $b_n(t)$ is inessential - it can be eliminated from the system (7.12). Thus only variable $u_n(t)$ is considered as a "true" Toda chain variable (for details see [57]).

Now we can construct the phase portrait for this variable. By the phase portrait we will assume the plot constructed on the plane x, y if one take points $P_0, P_1, P_2, \ldots$ with the coordinates $P_0 = (u_0, u_1)$, $P_1 = (u_1, u_2), \ldots, P_n = (u_n, u_{n+1})$. The phase portrait is an indicator of integrability: if the system is integrable the points $P_i, i = 0, 1, \ldots$ fill some smooth curve. Otherwise these points are distributed quasi-stochastically (so-called "stochastic web" [21]). In our case the variable $u_n(t)$ is given explicitly by (7.13), $u_{n+1}(t) = u_n(t - p/\omega)$ and hence the points $P_n = (u_n, u_{n+1})$ fill the symmetric biquadratic curve in its canonical form (3.24) by the proposition 9. The complex John algorithm in this biquadratic curve is equivalent to passing from the point P_n to point P_{n+1} and then to point P_{n+2}. Note that the time parameter t describes a smooth motion along this curve (Hamiltonian dynamics), whereas the John algorithm describes a "discrete motion". The periodic case is defined by a positive integer N such that $u_n(t) = u_{n+N}(t)$. From (7.13) it is seen that periodicity condition is equivalent to

$$pN = 2m_1\omega_1 + 2m_2\omega_2 \tag{7.14}$$

with some integers m_1, m_2. An important property of the elliptic Toda solutions is that the periodicity property (7.14) doesn't depend on the time parameter t, i.e. if periodic condition takes place for one value of t then it also holds for all others values of t. In terms of the (complex) John algorithm this means that a period of a point following the John mapping doesn't depend on choice of this initial point $P_0 = (u_0(t), u_1(t))$ on the curve.

Comparing obtained formulas for $u_n(t)$ with (4.15) and (4.16) we can say that the elliptic solutions of the Toda chain give an interesting illustration of the Poncelet theorem: periodic solutions (7.14) of the Toda chain correspond to periodicity of the Poncelet process on two parabolas.

7.3. **Elliptic grids in the theory of the rational interpolation.** In this section we describe briefly an interesting connection of the Poncelet problem with theory od so-called admissible grids for biorthogonal rational functions. This subject is important in theory of rational interpolations. For further details and relations with theory of special functions see, e.g. [42], [52], [53], [54], [63], [64].

Let us consider the set of rational functions $R_n(x)$ of the order $[n/n]$, which means that $R_n(x)$ are given by ratios of two n-th degree polynomials in x. In what follows it is assumed that all rational functions $R_n(x)$ have only simple poles.

We take $\alpha_1, \alpha_2, \dots, \alpha_n$ as n distinct prescribed positions of the poles of $R_n(x)$. Then $R_n(x)$ can be written as a sum of partial fractions

$$R_n(x) = t_0^{(n)} + \sum_{i=1}^{n} \frac{t_i^{(n)}}{x - \alpha_i} \tag{7.15}$$

with the coefficients $t_i^{(n)}$, $i = 1, 2, \dots, n$, playing the role of residues of $R_n(x)$ at the poles α_i. The coefficient $t_0^{(n)}$ can be interpreted as $\lim_{x\to\infty} R_n(x)$.

Let $x(s)$ be a "grid", i.e. a function in the argument s. We would like to construct so-called lowering operator $\mathcal{D}_{x(s)}$ in the space of rational functions $R_n(x)$ defined on this grid in the following way. We take as a definition of the lowering operator $\mathcal{D}_{x(s)}$ a divided difference operator in the parameterizing variable s, which obeys the following properties:

i) the grid $x(s)$ is a meromorphic function of $s \in \mathbb{C}$ which is invertible in some domain of the complex plane;

(ii) for any function $F(s)$ one has

$$\mathcal{D}_{x(s)} F(s) = \chi(s)(F(s+1) - F(s)),$$

where $\chi(s)$ is some function to be determined;

(iii) $\mathcal{D}_{x(s)} R_1(x) = const$, where $R_1(x)$ is an arbitrary rational function of the order $[1/1]$ with the only pole at $x = \alpha_1$;

(iv) the operator $\mathcal{D}_{x(s)}$ transforms any rational function $R_n(x)$ with the *prescribed* poles $\alpha_1, \dots, \alpha_n$ to a rational function $\tilde{R}_{n-1}(y(s))$ of the adjacent grid $y(s)$ with some other poles $\beta_1, \dots, \beta_{n-1}$;

(v) the operator $\mathcal{D}_{x(s)}$ is "transitive": for any nonnegative integer j the operator $\mathcal{D}^{(j)}_{x(s)}$ defined as

$$\mathcal{D}^{(j)}_{x(s)} F(s) = \chi_j(s)(F(s+1) - F(s))$$

with some function $\chi_j(s)$ transforms any rational function $R_n(x)$ with the *prescribed* poles $\alpha_{j+1}, \dots, \alpha_{j+n}$ to a rational function $\tilde{R}_{n-1}(y(s))$ with one and the same adjacent grid $y(s)$ and the sequence of poles $\beta_{j+1}, \dots, \beta_{j+n-1}$;

(vi) we assume that the poles are nondegenerate: there are infinitely many distinct values of α_n and β_n and $\alpha_n \neq \alpha_{n+1}, \alpha_{n+2}$ and, similarly, $\beta_n \neq \beta_{n+1}, \beta_{n+2}$ for all n.

An important restriction is the condition of independence of $\mathcal{D}_{x(s)}$ on the order n of a rational function. The problem is to deduce the functions $\chi_j(s)$ and $x(s), y(s)$ from the given set of requirements.

From the properties (i)-(iii) we easily find

$$\chi(s) = \left(\frac{1}{x(s+1)-\alpha_1} - \frac{1}{x(s)-\alpha_1}\right)^{-1} = \frac{(x(s)-\alpha_1)(x(s+1)-\alpha_1)}{x(s)-x(s+1)}. \tag{7.16}$$

The function $\chi(s)$ is defined up to an inessential constant multiplier. Note that from (ii) we have $\mathcal{D}_{x(s)}R_0(z) = 0$.

The most non-trivial problem in the construction of the operator $\mathcal{D}_{x(s)}$ consists in establishing the properties (iv)-(v).

This problem was solved in [54]. It appears that the grid $x(s)$ as well as the grids $y(s), \alpha_s, \beta_s$ should belong to the class of the elliptic grids. This means that they should satisfy the biquadratic equation

$$A_1x^2(s+1)x^2(s) + A_2x^2(s)x(s+1) + A_3x^2(s+1)x(s) + A_4x(s+1)x(s) + A_5x^2(s+1) + A_6x^2(s) + A_7x(s+1) + A_8x(s) + A_9 = 0 \tag{7.17}$$

with some constants $A_i, i = 1, 2, \dots, 9$. As we already know this equation can be parameterized in terms of the elliptic functions and we thus arrive at elliptic grids $x(s)$. Thus the elliptic grids $x(s), y(s)$ are the most general ones to provide existence of the lowering operator for rational functions.

In the theory of the rational (sometimes called the Cauchy-Jacobi or Padé) interpolation these grids appear naturally for some class of self-similar solutions.

The Cauchy-Jacobi interpolation problem (CJIP) for the sequence Y_j of (nonzero) complex numbers can be formulated as follows [8], [44]. Given two nonnegative integers n, m, choose a system of (distinct) points $x_j, j = 0, 1, \dots, n+m$ on the complex plane. We are seeking polynomials $Q_m(x;n), P_n(x;m)$ of degrees m and n correspondingly such that

$$Y_j = \frac{Q_m(x_j;n)}{P_n(x_j;m)}, \quad j = 0, 1, \dots n+m \tag{7.18}$$

(in our notation we stress, e.g. that polynomial $Q_m(x;n)$, being degree m in x, depends on n as a parameter).

It can happens that solution of the CJIP doesn't exist. In this case it is reasonable to consider a *modified* CJIP:

$$Y_jP_n(x_j;m) - Q_m(x_j;n) = 0, \ j = 0, 1, \dots, n+m, \tag{7.19}$$

where polynomials $P_n(x;m), Q_m(x;n)$ can be now unrestricted. The problem (7.19) always has a nontrivial solution. In exceptional case, if the system (7.18) has no solutions, some zeroes of polynomials $P_n(z;m)$ and $Q_m(z;n)$ coincide with interpolated points x_s. Such points, in this case, are called unattainable [44].

The CJIP is called *normal* if polynomials $Q_m(x;n), P_n(x;m)$ exist for all values of $m, n = 0, 1, \ldots$ and polynomials $Q_m(x;n), P_n(x;m)$ have no common zeroes. This means, in particular, that polynomials $Q_m(x;n), P_n(x;m)$ have no roots, coinciding with interpolation points, i.e.

$$Q_m(x_j;n) \neq 0,\ P_n(x_j;m) \neq 0, \quad j = 0, 1, \ldots n+m \tag{7.20}$$

In a special case when there exists an analytic function $f(z)$ of complex variable such that $f(x_j) = Y_j$ the corresponding CJIP is called multipoint Padé approximation problem [8].

It is possible to show that the Cauchy-Jacobi interpolation problem is equivalent to theory of biorthogonal rational functions [64]. The elliptic solutions (obtained first in [52]) of this problem appear naturally if the interpolated function $f(z)$ satisfy the so-called discrete Riccati equation [42]. Geometric interpretation of obtained elliptic grids and their connection with the Poncelet problem can be found in [42] and [54].

Note that if terms of degree > 2 are absent in (7.17) (i.e. $A_1 = A_2 = A_3 = 0$) then corresponding grid is degenerated to so-called Askey-Wilson grid [7] which is the most general grid for orthogonal polynomials satisfying a linear second-order difference equation [61]. From geometrical point of view the Askey-Wilson grids correspond to the John algorithm for the second-degree curves (i.e. ellipsis, hyperbola or parabola) [42].

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